

THE CLASSIFICATION OF ISOTRIVIAALLY FIBRED SURFACES WITH

$$p_g = q = 2$$

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ABSTRACT. An isotrivially fibred surface is a smooth projective surface endowed with a morphism onto a curve such that all the smooth fibres are isomorphic to each other. The first goal of this paper is to classify the isotrivially fibred surfaces with $p_g = q = 2$ completing and extending a result of Zucconi. As an important byproduct, we provide new examples of minimal surfaces of general type with $p_g = q = 2$ and $K^2 = 4, 5$ and a first example with $K^2 = 6$.

1. INTRODUCTION

The classification of smooth connected minimal complex projective surfaces of general type with small invariants is far from being achieved, and up to now a complete classification seems out of reach. This is a reason why one first tries to understand and classify surfaces with particularly small invariants, for example with $\chi(\mathcal{O}_S) = 1$. If this is the case one has $1 = \chi(\mathcal{O}_S) = 1 - q + p_g$ and it follows that $p_g = q$. If we also assume that the surface is irregular (i.e., $q > 0$) then the Bogomolov-Miyaoka-Yau and Debarre inequalities, $K_S^2 \leq 9$, $K_S^2 \geq 2p_g$, imply $1 \leq p_g \leq 4$. If $p_g = q = 4$ we have a product of curves of genus 2, as shown by Beauville, while the case $p_g = q = 3$ was understood through the work of several authors [CCML, HP, Pi]. It seems that, the classification becomes more complicated as the value of p_g decreases. In this paper we address the case $p_g = q = 2$.

We say that a surface S is *isogenous to a product of curves* if $S = (C \times F)/G$, for C and F smooth curves and G a finite group acting freely on $C \times F$. Surfaces isogenous to a product were introduced by Catanese in [C1]. They are of general type if and only if both $g(C)$ and $g(F)$ are greater than or equal to 2 and in this case S admits a unique minimal realization where they are as small as possible. From now on, we tacitly assume that such a realization is chosen, so that the genera of the curves and the group G are invariants of S . We have two cases: the *mixed* one, where there exists some element in G exchanging the two factors (in this situation C and F must be isomorphic) and the *unmixed* one, where G acts faithfully on both C and F and diagonally on their product. A special case of surfaces isogenous to a product of unmixed type is the case of *generalized hyperelliptic surfaces* where G acts freely on C and $F/G \cong \mathbb{P}^1$.

A generalization of the unmixed case is the following: consider a finite group G acting faithfully on two smooth projective curves C and F of genus ≥ 2 , and diagonally, but not necessarily freely, on their product, and take the minimal resolution $S' \rightarrow T := (C \times F)/G$ of the singularities of T . In this case the holomorphic map:

$$f'_1: S' \longrightarrow C' := C/G$$

is called a *standard isotrivial fibration* if it is a relatively minimal fibration. More generally an *isotrivial fibration* is a fibration $f: S \rightarrow B$ from a smooth surface onto a smooth curve such that all the smooth fibres are isomorphic to each other. A monodromy argument shows that, in case the general fibre F is irrational, there is a birational realization of S as a quotient of a product of two curves $S \stackrel{\text{bir}}{\sim} (C \times F)/G \rightarrow C/G \cong B$.

Among the surfaces which admit an isotrivial fibration one can find examples of surfaces with $\chi(\mathcal{O}_S) = 1$. Since [C1] appeared several authors studied intensively standard isotrivially fibred

surfaces, and eventually classified all those, which are minimal with $p_g = q = 0$ [BCG, BCGP, BP] and with $p_g = q = 1$ [P2, CP, MP].

In this paper we complete the classification of isotrivially fibred surfaces with $p_g = q = 2$ (which was partially given in [Z]). Moreover we give a precise description of the corresponding locus in the moduli space of surfaces of general type. Indeed, by the results of [C1], this locus is a union of connected components in the case of surfaces isogenous to a product of curves, and irreducible subvarieties in the case of only isotrivial surfaces. We calculate the number of these components (subvarieties) and their dimensions. The following theorem summarizes our classification.

Theorem 1.1. *Let S be a minimal surface of general type with $p_g = q = 2$ such that it is either a surface isogenous to a product of curves of mixed type or it admits an isotrivial fibration. Let $\alpha: S \rightarrow \text{Alb}(S)$ be the Albanese map. Then we have the following possibilities:*

- (1) *If $\dim(\alpha(S)) = 1$, then $S \cong (C \times F)/G$ and it is generalized hyperelliptic. The classification of these surfaces is given by the cases labelled with *GH* in Table 1, where we specify the only possibilities for the genera of the two curves C and F , and for the group G .*
- (2) *If $\dim(\alpha(S)) = 2$, then there are three cases:*
 - *S is isogenous to product of unmixed type $(C \times F)/G$, and the classification of these surfaces is given by the cases labelled with *UnMix* in Table 1;*
 - *S is isogenous to a product of mixed type $(C \times C)/G$, there is only one case and it is given in Table 1 labelled with *Mix*;*
 - *$S \rightarrow T := (C \times F)/G$ is a minimal desingularization of T , and these surfaces are classified in Table 2.*

Type	K_S^2	$g(F)$	$g(C)$	G	IdSmallGroup	\mathbf{m}	\dim	n
GH	8	2	3	\mathbb{Z}_2	$G(2,1)$	(2^6)	6	1
GH	8	2	4	\mathbb{Z}_3	$G(3,1)$	(3^4)	4	1
GH	8	2	5	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$G(4,2)$	(2^5)	5	2
GH	8	2	5	\mathbb{Z}_4	$G(4,1)$	$(2^2, 4^2)$	4	1
GH	8	2	6	\mathbb{Z}_5	$G(5,1)$	(5^3)	3	1
GH	8	2	7	\mathbb{Z}_6	$G(6,2)$	$(2^2, 3^2)$	4	1
GH	8	2	7	\mathbb{Z}_6	$G(6,2)$	$(3, 6^2)$	3	1
GH	8	2	9	\mathbb{Z}_8	$G(8,1)$	$(2, 8^2)$	3	1
GH	8	2	11	\mathbb{Z}_{10}	$G(10,2)$	$(2, 5, 10)$	3	1
GH	8	2	13	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$G(12,5)$	$(2, 6^2)$	3	2
GH	8	2	7	S_3	$G(6,1)$	$(2^2, 3^2)$	4	1
GH	8	2	9	Q_8	$G(8,4)$	(4^3)	3	1
GH	8	2	9	D_4	$G(8,3)$	$(2^3, 4)$	4	2
GH	8	2	13	D_6	$G(12,4)$	$(2^3, 3)$	3	2
GH	8	2	13	$D_{4,3,-1}$	$G(12,1)$	$(3, 4^2)$	3	1
GH	8	2	17	$D_{2,8,3}$	$G(16,8)$	$(2, 4, 8)$	3	1
GH	8	2	25	$\mathbb{Z}_2 \times ((\mathbb{Z}_2)^2 \times \mathbb{Z}_3)$	$G(24,8)$	$(2, 4, 6)$	3	2
GH	8	2	25	$\text{SL}(2, \mathbb{F}_3)$	$G(24,3)$	$(3^2, 4)$	3	1
GH	8	2	49	$\text{GL}(2, \mathbb{F}_3)$	$G(48,29)$	$(2, 3, 8)$	3	1
UnMix	8	3	3	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$G(4,2)$	$(2^2), (2^2)$	4	1
UnMix	8	3	4	S_3	$G(6,1)$	$(3), (2^2)$	3	1
UnMix	8	3	5	D_4	$G(8,3)$	$(2), (2^2)$	3	1
Mix	8	3	3	\mathbb{Z}_4	$G(4,1)$	-	3	1

Table 1.

In Table 1 and 2 IdSmallGroup denotes the label of the group G in the GAP4 database of small groups, \mathbf{m} is the branching data. In Table 1 each item provides a union of connected components

of the moduli space of surfaces of general type, their dimension is listed in the column \dim and n is the number of connected components.

K_S^2	$g(C)$	$g(F)$	G	$IdSmallGroup$	m	Type	Num. Sing.	\dim	n
4	2	2	\mathbb{Z}_2	$G(2,1)$	$(2^2) (2^2)$	$\frac{1}{2}(1,1)$	4	4	1
4	3	3	D_4	$G(8,3)$	$(2) (2)$	$\frac{1}{2}(1,1)$	4	2	1
4	3	3	Q_8	$G(8,4)$	$(2) (2)$	$\frac{1}{2}(1,1)$	4	2	1
5	3	3	S_3	$G(6,1)$	$(3) (3)$	$\frac{1}{3}(1,1) + \frac{1}{3}(1,2)$	2	2	1
6	4	4	A_4	$G(12,3)$	$(2) (2)$	$\frac{1}{2}(1,1)$	2	2	1

Table 2.

In Table 2 each item provides a union of irreducible subvarieties of the moduli space of surfaces of general type, their dimension is listed in the column \dim and n is the number of subvarieties. Moreover the columns of Table 2 labelled with Type and Num. Sing. indicate the types and the number of singularities of T .

We point out that in Table 2 there are new examples of minimal surfaces of general type with $p_g = q = 2$ and $K_S^2 = 4, 5$, and a first example with $K_S^2 = 6$. It will be interesting to find, if there are any, examples of surfaces with $p_g = q = 2$ and $K_S^2 = 7$ or 9 .

We recall that surfaces of general type with $p_g = q = 2$ and $K_S^2 = 4$ were studied by Ciliberto and Mendes Lopes. Indeed they proved that the surfaces with $p_g = q = 2$ and non-birational bicanonical map are double coverings of a principally polarized abelian surfaces branched on a divisor $D \in |2\Theta|$, and they have $K_S^2 = 4$ ([CML]). While Chen and Hacon ([CH]) constructed a first example of a surface with $K_S^2 = 5$.

The classification of the surfaces isogenous to a product involves techniques from both geometry and combinatorial group theory (developed in [BC, BCG, P1, CP]). In the second section of this paper we recall the relation between coverings of Riemann surfaces and orbifold surface groups, which enables us to transform the geometric problem of classification into an algebraic one.

In the third section we recall the notion of generalized hyperelliptic surface, and we shall see, following [C1] and [Z], that all the surfaces with $p_g = q = 2$ and not of Albanese general type are generalized hyperelliptic. Using this fact and the material of section two we classify all such surfaces. We notice that such classification was partially given in [Z] using different techniques. We proceed then to classify the surfaces isogenous to a product of unmixed type and of Albanese general type and finally we study the mixed case.

In the fourth section we consider the case when $(C \times F)/G$ is singular. We provide there new examples of surfaces with $p_g = q = 2$.

In the fifth section we treat the problem of the description of the moduli spaces. We recall a theorem of Bauer and Catanese (Theorem 1.3 [BC]) which tells how to calculate the number of the connected components of the moduli space. Moreover we calculate the action of the mapping class group of a Riemann surface of genus 2 on its fundamental group. In the end we shall see that calculating the number of connected components is a task which cannot be achieved easily without using a computer.

In the sixth section we calculate the fundamental groups of the found isotrivial surfaces. We recall in that section two structure theorems one for the fundamental group of surfaces isogenous to a product of curves and one for the fundamental group of isotrivial surfaces following [BCGP].

In the appendix written by S. Rollenske, it is described the GAP4 program we wrote to finish our classification and it is explained how to use it. Since it is written in great generality we hope it can be used for other tasks. For example we were able to finish the classification given in [CP] adding the number of connected components of the moduli spaces. We tested the program also on the cases with $p_g = q = 0$ given in [BCG] and $p_q = q = 1$ given in [P1].

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Notation and conventions. We shall denote by S a smooth, irreducible, complex, projective surface. We shall also use the standard notation in surface theory, hence we denote by Ω_S^p the sheaf of holomorphic p -forms on S , $p_g := h^0(S, \Omega_S^2)$ the *geometric genus* of S , $q := h^0(S, \Omega_S^1)$ the *irregularity* of S , $\chi(S) = \chi(\mathcal{O}_S) = 1 + p_g - q$ the *holomorphic Euler-Poincaré characteristic*, $e(S)$ the *topological Euler number*, and K_S^2 the *self-intersection of the canonical divisor* (see e.g., [Bad, BHPV, B]). Moreover, if C is a smooth compact complex curve (Riemann surface), then $g(C)$ will denote its *genus*.

We shall also use a standard notation in group theory, hence we denote by \mathbb{Z}_n the *cyclic group* of order n , by A_n the *alternating group* on n letters, by S_n the *symmetric group* on n letters, by D_n the *dihedral group* of order $2n$, by Q_8 the *group of quaternions*, by $D_{p,q,r}$ a group with following presentation $\langle x, y \mid x^p = y^q = 1, xyx^{-1} = y^r \rangle$ and $(r, q) = 1$, by $\text{GL}(2, q)$ the group of invertible 2×2 matrices over the finite field with q elements, which we denote by \mathbb{F}_q , and by $\text{SL}(2, q)$ the subgroup of $\text{GL}(2, q)$ comprising the matrices with determinant 1. With $Z(G)$ we shall denote the center of a group G , moreover let $H \leq G$ be a subgroup then the normalizer of H in G it will be denoted by $N_G(H)$, while by $C_G(x)$ we denote the centralizer of $x \in G$. In addition we shall write $x \sim_G y$ if $x, y \in G$ are conjugate to each other.

2. GROUP THEORETICAL PRELIMINARIES

The study of surfaces isogenous to a product of curves is strictly linked with the study of Galois coverings of Riemann surfaces with fixed branching data. We collect in this section some standard facts on coverings of Riemann surfaces from an algebraic point of view following the notation of [P1] and [C2].

Definition 2.1. Let G be a finite group and let:

$$0 \leq g', \quad 2 \leq m_1 \leq \dots \leq m_r$$

be integers. A generating vector for G of type $(g' \mid m_1, \dots, m_r)$ is a $(2g' + r)$ -tuple of elements of G :

$$\mathcal{V} = \{a_1, b_1, \dots, a_{g'}, b_{g'}, c_1, \dots, c_r\}$$

such that the following are satisfied:

- (1) The set \mathcal{V} generates G .
- (2) Denoting by $|c|$ the order of c :
A: $|c_i| = m_i \quad \forall 1 \leq i \leq r$, or
B: there exist a permutation $\sigma \in S_r$ such that:

$$|c_1| = m_{\sigma(1)}, \dots, |c_r| = m_{\sigma(r)},$$

- (3) $\prod_{i=1}^{g'} [a_i, b_i] c_1 \dots c_r = 1$.

If such a \mathcal{V} exists we say that G is $(g' \mid m_1, \dots, m_r)$ -generated. We refer to $\mathbf{m} := m_1, \dots, m_r$ as the branching data.

Moreover if $g' = 0$ a generating vector is said to be spherical.

When we consider the definition with (ii) **A** we shall clearly speak of *ordered vectors*. *Unordered vectors* ((ii) **B**) are needed only when we tackle the problem of the moduli space, and so until the last section we shall suppose that the generating vectors are all ordered. We shall also use the notation, for example, $(g' \mid 2^4, 3^2)$ to indicate the tuple $(g' \mid 2, 2, 2, 2, 3, 3)$.

We have the following reformulation of the Riemann Existence Theorem.

Proposition 2.2. *A finite group G acts as a group of automorphisms of some compact Riemann surface C of genus g if and only if there exist integers $g' \geq 0$ and $m_r \geq m_{r-1} \geq \dots \geq m_1 \geq 2$ such that G is $(g' \mid m_1, \dots, m_r)$ -generated for some generating vector $\{a_1, b_1, \dots, a_{g'}, b_{g'}, c_1, \dots, c_r\}$, and the following Riemann-Hurwitz relation holds:*

$$(1) \quad 2g - 2 = |G|(2g' - 2 + \sum_{i=1}^r (1 - \frac{1}{m_i})).$$

If this is the case, then g' is the genus of the quotient Riemann surface $C' := C/G$ and the Galois covering $C \rightarrow C'$ is branched in r points p_1, \dots, p_r with branching numbers m_1, \dots, m_r respectively. Moreover if $r = 0$ the covering is said to be *unramified* or *étale*.

One introduces the following abstract group.

Definition 2.3. *Let us denote by $\Gamma = \Gamma(g' \mid m_1, \dots, m_r)$ the abstract group with presentation:*

$$\Gamma(g' \mid m_1, \dots, m_r) := \langle \alpha_1, \beta_1, \dots, \alpha_{g'}, \beta_{g'}, \gamma_1, \dots, \gamma_r \mid \gamma_1^{m_1} = \dots = \gamma_r^{m_r} = \prod_{k=1}^{g'} [\alpha_k, \beta_k] \gamma_1 \cdot \dots \cdot \gamma_r = 1 \rangle.$$

We shall call this group an orbifold surface group, following [C2].

Notice that other authors call this a *Fuchsian type group* of signature $(g' \mid m_1, \dots, m_r)$, see e.g., [Br]. Indeed, one can reinterpret the Riemann Existence Theorem in terms of exact sequence of groups via the Uniformization Theorem.

Proposition 2.4. *A finite group G acts as a group of automorphisms on some compact Riemann surface C of genus $g \geq 2$ if and only if there exists an exact sequence of groups:*

$$1 \longrightarrow \Pi_g \longrightarrow \Gamma \xrightarrow{\theta} G \longrightarrow 1,$$

where $\Gamma = \Gamma(g' \mid m_1, \dots, m_r)$ is an orbifold surface group, and Π_g is the fundamental group of C .

Since Π_g is torsion free, it follows that the order of each generators $\gamma_i \in \Gamma$ is the same as the one of $\theta(\gamma_i) \in G$ (see [Br, Lemma 3.6]). This is a reason to give the following definition.

Definition 2.5. *Let G be a finite group. An epimorphism $\theta : \Gamma = \Gamma(g' \mid m_1, \dots, m_r) \rightarrow G$ is called admissible if $\theta(\gamma_i)$ has order m_i for all i . If an admissible epimorphism exists, then clearly G is $(g' \mid m_1, \dots, m_r)$ -generated.*

One notices that the image of the generators of an orbifold surface group under an admissible epimorphism are exactly the generating vectors of Definition 2.1.

Thanks to Propositions 2.2 we are able to translate the geometrical problem of finding Galois coverings $C \rightarrow C/G$ into the algebraic problem of finding generating vectors for G of type $(g(C/G) \mid m_1, \dots, m_r)$.

3. SURFACES ISOGENOUS TO A PRODUCT WITH $p_g = q = 2$

Recall the following definitions.

Definition 3.1. *A surface S is said to be of Albanese general type, if $\dim(\alpha(S)) = 2$, where $\alpha : S \rightarrow \text{Alb}(S)$ is the Albanese map.*

Notice that if $q(S) = 2$ then $\dim(\text{Alb}(S)) = 2$, thus the Albanese map is surjective if and only if S is of Albanese of general type.

Definition 3.2. *A surface S is said to be isogenous to a (higher) product of curves if and only if, equivalently, either:*

- (1) *S admits a finite unramified covering, which is isomorphic to a product of curves of genera at least two, or*

- (2) S is a quotient $S := (C \times F)/G$ where C, F are curves of genus at least two, and G is a finite group acting freely on $C \times F$.

By Proposition 3.11 of [C1] the two properties (i) and (ii) are equivalent. Using the same notation as in definition 3.2, let S be a surface isogenous to a product, and $G^\circ := G \cap (Aut(C) \times Aut(F))$. Then G° acts on the two factors C, F and diagonally on the product $C \times F$. If G° acts faithfully on both curves, we say that $S = (C \times F)/G$ is a *minimal realization*. In [C1] is also proven that any surface isogenous to a product admits a unique minimal realization. From now on we shall work only with minimal realization.

By [C1, Lemma 3.8] there are two cases: the *mixed* case where the action of G exchanges the two factors, in this case C and F are isomorphic and $G^\circ \neq G$; and the *unmixed* case, where $G = G^\circ$ and therefore it acts diagonally.

We observe that a surface isogenous to a product of curves is of general type, it is always minimal (see [C1] Remark 3.2), and its numerical invariants are explicitly given in terms of the genera of the curves and the order of the group by the following proposition.

Proposition 3.3 ([C1], Theorem 3.4). *Let $S = (C \times F)/G$ be a surface isogenous to a product and denote by d the order of G , then:*

$$(2) \quad e(S) = \frac{4(g(C) - 1)(g(F) - 1)}{d},$$

$$(3) \quad K_S^2 = \frac{8(g(C) - 1)(g(F) - 1)}{d},$$

$$(4) \quad \chi(S) = \frac{(g(C) - 1)(g(F) - 1)}{d}.$$

From now on let S be a surface of general type with $p_g = q = 2$ and not of Albanese general type. According to [C1] we give the following definition.

Definition 3.4. *A surface isogenous to a product of unmixed type $S := (C \times F)/G$ is said to be generalized hyperelliptic if:*

- (1) *the Galois covering $C \rightarrow C/G$ is unramified,*
- (2) *the quotient curve F/G is isomorphic to \mathbb{P}^1 .*

The following theorem gives a characterization of generalized hyperelliptic surfaces.

Theorem 3.5 ([C1], Theorem 3.18). *Let S be a surface such that:*

- (1) $K_S^2 = 8\chi(\mathcal{O}_S) > 0$
- (2) S has irregularity $q \geq 2$ and the Albanese map is a pencil.

Then, letting g be the genus of the Albanese fibres, we have: $(g - 1) \leq \frac{\chi(\mathcal{O}_S)}{q-1}$.

A surface S is generalized hyperelliptic if and only if (i) and (ii) hold and $g = 1 + \frac{\chi(\mathcal{O}_S)}{q-1}$.

In particular, every S satisfying (i) and (ii) with $p_g = 2q - 2$ is generalized hyperelliptic where G (from the definition) is a group of automorphisms of the curve F of genus 2 with $F/G \cong \mathbb{P}^1$.

Corollary 3.6 (Zucconi, [Z] Prop. 4.2). *If S be a surface of general type with $p_g = q = 2$ and not of Albanese general type. Then S is generalized hyperelliptic.*

Remark 3.7. We collect all the properties of a surface S of general type with $p_g = q = 2$ not of Albanese general type. Let $\alpha : S \rightarrow Alb(S)$ the Albanese map and $B := \alpha(S)$. Then:

- (1) S is isogenous to an unmixed product $(C \times F)/G$.
- (2) $K_S^2 = 8$.
- (3) $g(F) = 2$ and $F/G \cong \mathbb{P}^1$.
- (4) $C \rightarrow C/G$ is unramified and $C/G \cong B$ has genus 2.
- (5) $|G| = (g(C) - 1)(g(F) - 1) = (g(C) - 1)$.

To classify all the groups and the genera of smooth curves of surfaces isogenous to a product with $p_g = q = 2$ and not of Albanese general type one can proceed as follows: first one classifies all possible finite groups G which induce a Galois covering $f: F \rightarrow \mathbb{P}^1$ with $g(F) = 2$, second one has to check whether such groups G induce an unramified Galois covering $g: C \rightarrow B \cong C/G$, where the genus of B is 2 and the genus of C is determined by the Riemann-Hurwitz formula.

We notice that the action of G on the product $C \times F$ is always free, since the action on C is free.

Theorem 3.8. *Let S be a complex surface of general type with $p_g = q = 2$ not of Albanese general type. Then $S = (C \times F)/G$ is generalized hyperelliptic and assuming w.l.o.g. $g(F) = 2$ the only possibilities for the genus of C , the group G and the branching data \mathbf{m} for $F \rightarrow F/G \cong \mathbb{P}^1$ are given by the entries in Table 1 of Theorem 1.1 labelled with GH.*

Proof. We are in the hypothesis of Corollary 3.6, thus $S = (C \times F)/G$, where F , C and G have the property indicated in Remark 3.7.

The classification of the automorphisms groups of a Riemann surface F of genus 2 was given by Bolza in [Bol], moreover the classification of all the groups G acting effectively as a group of automorphisms of F such that the quotient F/G is isomorphic to \mathbb{P}^1 is given in [Z] or [Bro]. We give a full proof of the classification of the latter groups, since we are interested in obtaining a complete information including also the branching data.

By the Riemann-Hurwitz formula (1) applied to F we obtain:

$$(5) \quad 2g(F) - 2 = |G| \left(-2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i} \right) \right),$$

remembering that $g(F) = 2$ we get:

$$(6) \quad 2 = |G| \left(-2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i} \right) \right)$$

which yields:

$$|G| \left(\frac{r}{2} - 2 \right) \leq 2 \leq |G|(r - 2),$$

and since $|G| \geq 2$ we have $3 \leq r \leq 6$.

We examine all the cases proceeding as follows: for each r , using the fact that $2 \leq m_1 \leq \dots \leq m_r$, and by (6), we can bound the order of G from above by a rational function of m_1 and from below by m_1 , since m_1 divides $|G|$, and we analyze case by case. As soon as m_1 gives a void condition, we repeat the same analysis using m_2 , and so on for all m_i 's. In the first case we shall perform a full calculation as an example.

Case $r = 6$.

In this case by (6) we have :

$$2 = |G| \left(-2 + \sum_{i=1}^6 \left(1 - \frac{1}{m_i} \right) \right) \geq |G| \left(-2 + 6 \left(1 - \frac{1}{m_1} \right) \right),$$

which yields

$$m_1 \leq |G| \leq \frac{m_1}{2m_1 - 3},$$

then $m_1 = 2$ and $|G| = 2$. Therefore by equation (4) $g(C) = 3$, and since m_i divides $|G|$ for all $i = 1, \dots, 6$, we have $m_i = 2$ for all $i = 1, \dots, 6$. Then $G = \mathbb{Z}_2$, since it is $(0 \mid 2^6)$ -generated. We recover the first case in Table 1, i.e., $g(F) = 2$ $g(C) = 3$ and $\mathbf{m} = (2^6)$.

Notice that to fully recover this first case we still have to prove that \mathbb{Z}_2 induces an unramified covering $g: C \rightarrow B \cong C/\mathbb{Z}_2$, where the genus of B is 2, but this is obvious. From now on in order to avoid many repetitions we investigate the branching data and the order of the groups and it will be clear which case in Table 1 is recovered. Moreover we shall prove at the end that all the groups, we have found, induce an unramified covering $C \rightarrow C/G$ with quotient a curve of genus 2.

Case $r = 5$.

Proceeding as in the previous case, we have:

$$2 \leq |G| \leq 4.$$

If $|G| = 2$ then $m_i = 2$ for all $i = 1, \dots, 5$ which yields a contradiction to (6).

If $|G| = 3$ then $m_i = 3$ for all $i = 1, \dots, 5$ and again we have a contradiction.

If $|G| = 4$ then $m_i = 2$ for all $i = 1, \dots, 5$. Since the elements of order 2 generate the group we have $G = (\mathbb{Z}_2)^2$. Indeed we have $c_1, \dots, c_5 \in (\mathbb{Z}_2)^2 \setminus \{(0, 0)\}$ such that $\sum_{i=1}^5 c_i = 0$, for example take c_1, c_2 and c_3 all different from each other, and $c_1 = c_4 = c_5$.

Case $r = 4$.

In this case we have:

$$m_1 \leq |G| \leq \frac{2m_1}{2m_1 - 4},$$

then $m_1 \leq 3$.

If $m_1 = 3$, then $|G| = 3$ and $m_i = 3$ for all $i = 1, \dots, 4$. Clearly $G = \mathbb{Z}_3$ which is $(0 \mid 3^4)$ -generated, consider for example $c_1 = 1, c_2 = 1, c_3 = 2, c_4 = 2$.

Now suppose $m_1 = 2$, this gives no upper bound for the order of G , therefore looking at the possible values of m_2 , we have:

$$\text{l.c.m.}(2, m_2) \leq |G| \leq \frac{4m_2}{3m_2 - 6}.$$

We can exclude the cases with $m_2 \geq 3$, so $m_2 = 2$.

If we proceed further and look at the values of m_3 once $m_1 = m_2 = 2$, we have:

$$\text{l.c.m.}(2, m_3) \leq |G| \leq \frac{2m_3}{m_3 - 2},$$

so that $m_3 \leq 4$.

If $m_3 = 4$, then $|G| = 4, m_4 = 4$ and $G = \mathbb{Z}_4$, which is $(0 \mid 2^2, 4^2)$ -generated, take for example $c_1 = 2, c_2 = 2, c_3 = 1, c_4 = 3$.

If $m_3 = 3$, then $|G| = 6$ and we have $m_4 = 3$. We have two possibilities either $G = \mathbb{Z}_6$ or $G = S_3$, and both cases occur, since both groups are $(0 \mid 2, 2, 3, 3)$ -generated. For the first case consider for example $c_1 = 3, c_2 = 3, c_3 = 2$, and $c_4 = 4$, while for the latter one $c_1 = (1, 2), c_2 = (2, 3), c_3 = (1, 3, 2), c_4 = (1, 3, 2)$.

Let us consider the case $m_3 = 2$, then we have to look at the possible values of m_4 , since:

$$\text{l.c.m.}(2, m_4) \leq |G| = \frac{4m_4}{m_4 - 2}$$

then only possibilities are the following:

If $m_4 = 6$, then $|G| = 6$ and this case is impossible. Indeed G cannot be S_3 , since S_3 has no element of order 6. In addition let c_1, \dots, c_4 be the generators of order m_1, \dots, m_4 , then we must have $c_1 + c_2 + c_3 + c_4 = 0$, then it cannot be \mathbb{Z}_6 since the only element of order two is 3 and 3 plus an element of order six is never 0.

If $m_4 = 4$ then $|G| = 8$ then $G = D_4$, for example with the following generators: $c_1 = y, c_2 = yx, c_3 = x^2, c_4 = x$, where y is a reflection and x a rotation. The group G cannot be $\mathbb{Z}_2 \times \mathbb{Z}_4$ or Q_8 , since the conditions $c_1 + c_2 + c_3 + c_4 = 0$, respectively $c_1 \cdot c_2 \cdot c_3 \cdot c_4 = 1$ are not satisfied. G cannot be \mathbb{Z}_8 , because it is not $(2, 4)$ generated. G cannot be $(\mathbb{Z}_2)^3$ since it does not have any element of order 4.

If $m_4 = 3$ then $|G| = 12$, G cannot be \mathbb{Z}_{12} or $\mathbb{Z}_2 \times \mathbb{Z}_6$ because of the condition $c_1 + \dots + c_4 = 0$, G cannot be $D_{3,4,-1} := \langle x, y \mid x^4 = y^3 = 1, xyx^{-1} = y^{-1} \rangle$ because $D_{3,4,-1}$ has only one element of order 2. The two remaining cases are D_6 or A_4 . If G has four 3-Sylow subgroups then $G = A_4$, impossible since the elements of order 2 are in the Klein subgroup while c_4 is not. In the other case $\mathbb{Z}_3 \subset G$ is normal, and there is an element of order 6. We recover the case $G = D_6$ with generating vector: $c_1 = y, c_2 = yx, c_3 = x^3, c_4 = x^2$.

Case $r = 3$.

This case is much more involved than the previous ones. We have

$$m_1 \leq |G| \leq \frac{2m_1}{m_1 - 3},$$

then after a short calculation one sees that $m_1 \leq 5$.

If $\mathbf{m}_1 = 5$ then $|G| = 5$ and $m_2 = m_3 = 5$. The only possibility is $G = \mathbb{Z}_5$ which is $(0 \mid 5^3)$ -generated, for example consider $c_1 = 1, c_2 = 2$ and $c_3 = 2$.

If $\mathbf{m}_1 = 4$ one has that $|G| = 8$ and $m_2 = m_3 = 4$, and the only group of order 8 which is $(0 \mid 4, 4, 4)$ -generated is $G = Q_8$, consider for example $c_1 = i, c_2 = j$ and $c_3 = -k$. Notice that the other groups of order 8 containing an element of order 4 are $\mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_4$, and D_4 . The elements of order 4 in \mathbb{Z}_8 and in D_4 form proper subgroups, so these cases are excluded. In $\mathbb{Z}_2 \times \mathbb{Z}_4$ there are four elements of order 4, the sum of any two of them is an element of order at most 2, so condition $c_1 + c_2 + c_3 = 0$ cannot be satisfied.

If $\mathbf{m}_1 = 3$ we have to look at all possible values of m_2 , since:

$$\text{l.c.m.}(3, m_2) \leq |G| \leq \frac{3m_2}{m_2 - 3}$$

then only possibilities are the following:

If $m_2 = 6$, then $|G| = 6$ and $m_3 = 6$. The only possibility is $G = \mathbb{Z}_6$, consider for example as generators $c_1 = 4, c_2 = 1$ and $c_3 = 1$. Notice that S_3 has no elements of order 6.

If $m_2 = 4$, then $|G| = 12$ and $m_3 = 4$. In this case the only group of order 12 which can be generated by elements c_1, c_2, c_3 of order 3, 4, 4 and such that these elements satisfy

$$(7) \quad c_1 \cdot c_2 \cdot c_3 = 1 \text{ (or additively } c_1 + c_2 + c_3 = 0),$$

is $D_{4,3,-1}$, choose for example $c_1 = y, c_2 = xy$ and $c_3 = x^3$, where the notation is the one given above. Notice that all the other groups either do not have an element of order 4 or it is \mathbb{Z}_{12} , which fails condition (7).

In the case $m_2 = 3$ we have to look at the possible values of m_3 , since:

$$\text{l.c.m.}(3, m_3) \leq |G| \leq \frac{6m_3}{m_3 - 3},$$

then the only possibilities are the following:

If $m_3 = 9$, then $|G| = 9$ and G could be only \mathbb{Z}_9 , but condition (7) is not satisfied by elements of order 3, 3, 9, which excludes this case.

If $m_3 = 6$, then $|G| = 12$. Also this case has to be excluded because: A_4 does not have an element of order 6, while for D_6 and $(\mathbb{Z}_2)^2 \times \mathbb{Z}_3$ the elements of order 3 and 6 cannot generate, in the end for the last two groups (7) fails.

If $m_3 = 5$ then $|G| = 15$ and we have only \mathbb{Z}_{15} , but its generating elements of order 3 and 5 do not satisfy (7).

If $m_3 = 4$ then $|G| = 24$. Here the number of groups involved or their order can be considerably large. In order to avoid many repetitions if these numbers are excessively large, where indicated, we use a computer program in GAP4 (see Appendix for an explanation of the program), to check the corresponding cases. Indeed the computer shows that among the 15 groups of order 24 the only one which can be $(0 \mid 3, 3, 4)$ -generated is `SmallGroup(24,3)`, which corresponds to $G = \text{SL}(2, \mathbb{F}_3)$. This exhausts all the cases with $m_1 = 3$.

If $\mathbf{m}_1 = 2$ we look at all possible values of m_2 since:

$$\text{l.c.m.}(2, m_2) \leq |G| \leq \frac{4m_2}{m_2 - 4},$$

then only possibilities are the following:

If $m_2 = 8$, then $|G| = 8$ and $m_3 = 8$ which yields $G = \mathbb{Z}_8$, choose for example as generating vector $c_1 = 4, c_2 = 5, c_3 = 7$.

If $m_2 = 6$, then $|G| = 12$ and $m_3 = 6$ which yields the case: $G = \mathbb{Z}_2 \times \mathbb{Z}_6$, choose for example as generating vector $c_1 = (1, 3), c_2 = (1, 2), c_3 = (0, 1)$. Notice that it cannot be \mathbb{Z}_{12} because

of (7), G cannot be A_4 because it does not have any element of order 6. Moreover G cannot be D_6 , since to generate it one needs a reflection y but the condition $c_2c_3 = y$ can never hold since the only elements with order 6 are x and x^5 , with x rotation. Finally $D_{4,3,-1}$ is impossible because the two elements of order 6 and the element of order 2 do not satisfy (7).

If $m_2 = 5$ then $|G| \leq 20$, looking at the branching data the only two possible cases are $|G| = 20, 10$.

If $|G| = 20$, then $m_3 = 5$. Among the 5 groups of order 20 a computer computation shows that none of them are $(0 \mid 2, 5, 5)$ –generated.

If $|G| = 10$ then $m_3 = 10$, which gives $G = \mathbb{Z}_{10}$ for example if $c_1 = x^5$, $c_2 = 4$, $c_3 = 1$. G cannot be D_5 since it has no element of order 10.

If $m_2 \leq 4$ one has to look at all possible values of m_3 .

Let $m_2 = 4$, since:

$$\text{l.c.m.}(4, m_3) \leq |G| = \frac{8m_3}{m_3 - 4}$$

then the only possibilities are the following:

If $m_3 = 12$, then $|G| = 12$ and G could be only \mathbb{Z}_{12} , but condition (7) cannot be satisfied by elements of order 2, 4, 12, therefore this case is excluded.

If $m_3 = 8$, then $|G| = 16$, among the 14 groups of order 16 a GAP4 computation shows that only **SmallGroup**(16,8) (i.e., $G = D_{2,8,3}$) can be $(0 \mid 2, 4, 8)$ –generated.

If $m_3 = 6$, then $|G| = 24$ and the only possibility for G is **SmallGroup**(24,8), which is $G = \mathbb{Z}_2 \ltimes ((\mathbb{Z}_2)^2 \times \mathbb{Z}_3)$. This case was also accomplished using GAP4.

One sees that case $m_3 = 5$ is impossible, here again it is needed a computational fact: none of the 14 groups of order 40 can be $(0 \mid 2, 4, 5)$ generated.

We now consider the case $m_2 = 3$. We look at all possible values of m_3 , since:

$$\text{l.c.m.}(6, m_3) \leq |G| = \frac{12m_3}{m_3 - 6}$$

then only possibilities are the following.

If $m_3 = 18$, then $|G| = 18$ and $G = \mathbb{Z}_{18}$, but condition (7) cannot be satisfied by elements of order 2, 3, 18.

If $m_3 = 12$, then $|G| = 24$, a computer calculation shows that among the 15 groups of order 24 none of them are $(0 \mid 2, 3, 12)$ –generated.

We can also exclude the case $m_3 = 10$ because none of the groups of order 30 is $(0 \mid 2, 3, 10)$ –generated.

One can exclude the case $m_3 = 9$, because among the 14 groups of order 36 a computer computation shows that none of them are $(0 \mid 2, 3, 9)$ –generated.

Case $m_3 = 7$ is also excluded, though there are 15 groups of order 84, a computer computation shows that none of them can be $(0 \mid 2, 3, 7)$ –generated.

The remaining case is $m_3 = 8$ which gives $|G| = 48$. A computer computation shows that among the 52 groups of order 48 only **SmallGroup**(48,29) (i.e., $G = \text{GL}(2, \mathbb{F}_3)$) satisfies all the necessary conditions. This recovers the last case of Table 1.

Now we have to see whether for each possible group G there is a surjective homomorphism:

$$\Gamma(2 \mid -) \twoheadrightarrow G.$$

Indeed this is true in all the cases, more precisely one notices that all the possible groups G can be two generated, call the generators x and y . Then we have the following epimorphism from $\Gamma(2 \mid -)$ to G :

$$\alpha_1 \mapsto x, \quad \beta_1 \mapsto 1, \quad \alpha_2 \mapsto y, \quad \beta_2 \mapsto 1.$$

□

We have analyzed the case when the image of the Albanese map is a curve. Now we want to see whether there are surfaces isogenous to a product of unmixed type with $p_g = q = 2$ and of Albanese general type.

Let $S \rightarrow B$ an isotrivial fibration with general fibre F and $g(F) \geq 2$, then a monodromy argument shows that S is birational to the quotient of a product of two curves $(C \times F)/G$

modulo the diagonal action of a finite group G and $C/G \cong B$ (see e.g., [S, §1.1]). Moreover by [S, Proposition 2.2] we have:

$$(8) \quad q(S) = g(C/G) + g(F/G).$$

This applies in particular if S is a surface isogenous to a product of unmixed type. If $q(S) = 2$, there are two cases: either (w.l.o.g) $g(C/G) = 2$ and $g(F/G) = 0$, or $g(C/G) = 1$ and $g(F/G) = 1$. By [Z, Proposition 4.3] the first case is completely solved.

For the second case we search for surfaces isogenous to an unmixed product $S = (C \times F)/G$ such that C/G and F/G are both elliptic curves and $\chi(S) = 1$. We need the following two results to simplify our search.

Lemma 3.9. [Z, Lemma 2.3, Corollary 2.4] *Let S be surface of Albanese general type with $p_g = q = 2$. Let $\phi: S \rightarrow B$ be a fibration of curves of genus g . If the genus of B is $b > 0$, then $b = 1$ and $2 \leq g \leq 5$.*

Lemma 3.10. *If G is an abelian group and G is $(g' \mid m_1, \dots, m_r)$ -generated, then $r \neq 1$.*

Proof. Suppose G abelian and $r = 1$. Then the relation $\Pi_{i=1}^{g'} [a_i, b_i] c_1 = 1$ yields $c_1 = 1$. \square

In case the Albanese map is not surjective we have that S is of generalized hyperelliptic type, hence one of the two coverings is étale, and we have always a free action of G on the product $C \times F$. In case the Albanese map is surjective we do not have an étale covering, so we also have to check whether the action of G on the product of the two curves is free or not.

Remark 3.11. Let $\mathcal{V}_1 := (a_{1,1}, b_{1,1}, c_{1,1}, \dots, c_{1,r_1})$ and $\mathcal{V}_2 := (a_{2,1}, b_{2,1}, c_{2,1}, \dots, c_{2,r_2})$ be generating vectors of type $(1 \mid m_{i,1}, \dots, m_{i,r_i})$ for $i = 1, 2$ respectively for a finite group G . Then the cyclic groups $\langle c_{1,1} \rangle, \dots, \langle c_{1,r_1} \rangle$ and their conjugates provide the non-trivial stabilizers for the action of G on C , whereas $\langle c_{2,1} \rangle, \dots, \langle c_{2,r_2} \rangle$ and their conjugates provide the non-trivial stabilizers for the action of G on F . The singularities of $(C \times F)/G$ arise from the points of $C \times F$ with nontrivial stabilizer, since the action of G on $C \times F$ is diagonal, it follows that the set \mathcal{S} of all nontrivial stabilizer for the action of G on $C \times F$ is given by $\Sigma(\mathcal{V}_1) \cap \Sigma(\mathcal{V}_2)$ where

$$\Sigma(\mathcal{V}_i) := \bigcup_{h \in G} \bigcup_{j=0}^{\infty} \bigcup_{k=1}^{r_i} h \cdot c_{i,k}^j \cdot h^{-1}.$$

It is clear that if we want $(C \times F)/G$ to be smooth we have to require $\mathcal{S} = \{1_G\}$. If this is the case we shall say that \mathcal{V}_1 and \mathcal{V}_2 are *disjoint*.

Theorem 3.12. *Let S be a complex surface with $p_g = q = 2$ of Albanese general type and isogenous to a product of unmixed type. Then S is minimal of general type and the only possibilities for the genera of the two curves C, F , the group G and the branching data \mathbf{m} respectively for $F \rightarrow F/G$ and $C \rightarrow C/G$ are given by the entries in Table 1 labelled with *UnMix*.*

Proof. By Lemma 3.9 we have, that $2 \leq g(F) \leq 5$ and, w.l.o.g., $g(F) \leq g(C)$. Moreover we have $g(C/G) = g(F/G) = 1$. We analyze case by case:

$$\mathbf{g}(\mathbf{F}) = \mathbf{2}.$$

From

$$(9) \quad 2g(F) - 2 = |G| \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right)$$

and $\sum_{i=1}^r \left(1 - \frac{1}{m_i}\right) \geq \frac{1}{2}$ we have:

$$2 \leq |G| \leq 4,$$

which yields:

$|G| = 4$ if and only if $r = 1$ and $m_1 = 2$;

$|G| = 3$ if and only if $r = 1$ and $m_1 = 3$;

$|G| = 2$ if and only if $m_1 = m_2 = 2$.

The first two cases contradict Lemma 3.10, the third one is also impossible. First notice that,

from equation (4), $g(C) = 3$, and for F and C we have respectively the following branching data: $(2, 2)$ and $(2, 2, 2, 2)$. It follows that we do not have any free action of \mathbb{Z}_2 on $C \times F$.

$\mathbf{g}(\mathbf{F}) = 3$.

From equation (9) we have:

$$2 \leq |G| \leq 8,$$

moreover 2 divides $|G|$ by equation (4). Then we have to analyze the cases: $|G| = 8, 6, 4, 2$.

If $|G| = 8$ then $g(C) = 5$, and by Riemann-Hurwitz the branching data for F and C are respectively (2) and $(2, 2)$. By Lemma 3.10 G is not abelian and since it is $(1 \mid 2^2)$ -generated it must be D_4 . Indeed G cannot be Q_8 , because Q_8 is not $(1 \mid 2^2)$ -generated, since the only element of order 2 is -1 . One sees that D_4 acts on $C \times F$ freely, hence this case occurs. Indeed one can choose the following generating vectors:

$$a_{1,1} = x \quad b_{1,1} = y \quad c_{1,1} = x^2;$$

$$a_{2,1} = x \quad b_{2,1} = y \quad c_{2,1} = x^2 y \quad c_{2,2} = y,$$

where x is a rotation and y is a reflection. Then $\{x^2\}$ is one conjugacy class and $\{y, x^2 y\}$ is another, hence the two vectors are disjoint.

If $|G| = 6$ then $g(C) = 4$. The Riemann-Hurwitz formula yields (3) as branching data for F and $(2, 2)$ for C , which yield $G = S_3$. One sees that any pair of generating vectors is disjoint since the conjugacy classes of elements of order 2 and 3 are disjoint, thus S_3 acts on $C \times F$ freely, and this case occurs.

If $|G| = 4$ then $g(C) = 3$. In this case the branching data of F and C are respectively $(2, 2)$ and $(2, 2)$. If $G = \mathbb{Z}_4$ the action cannot be free, since there is only one element of order two. For $\mathbb{Z}_2 \times \mathbb{Z}_2$ there is a free action. We first notice, since G is abelian, $2c_1 = 2c_2 = c_1 + c_2 = 0$ and $2c'_1 = 2c'_2 = c'_1 + c'_2 = 0$, then we can choose $c = c_1 = c_2$ and $c' = c'_1 = c'_2$. If we choose, for example, $c = (1, 1)$ and $c' = (1, 0)$ we see that this case occurs.

We observe that the case $|G| = 2$ leads to a contradiction to $g(F) \leq g(C)$.

$\mathbf{g}(\mathbf{F}) = 4$.

From equation (9) we have:

$$3 \leq |G| \leq 12,$$

moreover 3 must divide $|G|$ by equation (4). Furthermore since we assumed $g(F) \leq g(C)$ the only remaining cases are $|G| = 12, 9$.

If $|G| = 12$ then $g(C) = 5$. The branching data of F and C are respectively (2) and (3) . There is no non-abelian group of order 12 which is simultaneously $(1 \mid 2)$ and $(1 \mid 3)$ -generated. To see this one notices that the derived subgroups of D_6 and $D_{3,4,-1}$ are both isomorphic to \mathbb{Z}_3 , hence in both cases there are no commutators of order 2, therefore the two groups cannot be $(1 \mid 2)$ -generated. Moreover A_4 is not $(1 \mid 3)$ -generated because its derived subgroup is $\mathbb{Z}_2 \times \mathbb{Z}_2$, hence there are no commutators of order 3.

If $|G| = 9$ then $g(C) = 4$. We see that the branching data for F is (3) , since all the groups of order 9 are abelian, this case does not occur.

$\mathbf{g}(\mathbf{F}) = 5$.

From equation (9) we have:

$$4 \leq |G| \leq 16,$$

moreover 4 must divide $|G|$, and since $g(F) \leq g(C)$ the only case remaining is $|G| = 16$.

If $|G| = 16$ then $g(C) = 5$, and the branching data for F and C are (2) and (2) . Looking at the table in [CP] one sees that among the 14 groups of order 16 only $\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^2$, $D_{4,4,-1}$ and $D_{2,8,5}$ are $(1 \mid 2)$ -generated. Moreover with a computer computation using the program of the Appendix, one sees that in these cases the action of the groups on the product $C \times F$ cannot be free. Therefore this case does not occur. \square

Now we study the mixed case.

Theorem 3.13. [C1, Proposition 3.16] *Assume that G° is a finite group satisfying the following properties:*

- (i) G° acts faithfully on a smooth curve C of genus $g(C) \geq 2$,
- (ii) There is a non split extension:

$$(10) \quad 1 \longrightarrow G^\circ \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 1.$$

Let us fix a lift $\tau' \in G$ of the generator of \mathbb{Z}_2 . Conjugation by τ' defines an element $[\varphi]$ of order ≤ 2 in $\text{Out}(G^\circ)$.

Let us choose a representative $\varphi \in \text{Aut}(G^\circ)$ and let $\tau \in G^\circ$ be such that φ^2 is equal to conjugation by τ . Denote by Σ_C the set of elements in G° fixing some point on C and assume that both the following conditions are satisfied:

m1: $\Sigma_C \cap \varphi(\Sigma_C) = \{1_{G^\circ}\}$

m2: for all $\gamma \in G^\circ$ we have $\varphi(\gamma)\tau\gamma \notin \Sigma_C$.

Then there exists a free, mixed action of G on $C \times C$, hence $S = (C \times C)/G$ is a surface of general type isogenous to a product of mixed type. More precisely, we have

$$\gamma(x, y) = (\gamma x, \varphi(\gamma)y) \text{ for } \gamma \in G^\circ$$

$$\tau'(x, y) = (y, \tau x).$$

Conversely, every surface of general type isogenous to a product of mixed type arises in this way.

Notice that G° is the subgroup of transformations not exchanging the two factors of $C \times C$.

Proposition 3.14. *Let $(C \times C)/G$ be a surface with $p_g = q = 2$ isogenous to a product of mixed type. Then $B = C/G^\circ$ is a curve of genus $g(B) = 2$.*

Proof. From Proposition 3.15 [C1] we have:

$$\begin{aligned} H^0(\Omega_S^1) &= (H^0(\Omega_C^1) \oplus H^0(\Omega_C^1))^G = (H^0(\Omega_C^1)^{G^\circ} \oplus H^0(\Omega_C^1)^{G^\circ})^{G/G^\circ} = \\ &= (H^0(\Omega_B^1) \oplus H^0(\Omega_B^1))^{G/G^\circ}. \end{aligned}$$

Since S is of mixed type, the quotient $G/G^\circ = \mathbb{Z}_2$ exchanges the last summands, hence $h^0(\Omega_S^1) = h^0(\Omega_B^1) = 2$. \square

Theorem 3.15. *Let S be a complex surface with $p_g = q = 2$ and isogenous to a product of mixed type. Then S is minimal of general type and the only possibility for the genus of the curve C and the group G is given by the entry Mix of Table 1, moreover G° acts freely on C .*

Proof. From the fact that $|G| = (g(C) - 1)^2$ and that G° is a subgroup of index 2 in G we have:

$$(11) \quad |G^\circ| = \frac{(g(C) - 1)^2}{2},$$

hence $g(C)$ must be odd.

Since $(C \times C)/G^\circ$ is isogenous to a product of unmixed type and by Proposition 3.14 we have $g(C/G^\circ) = 2$, the Riemann-Hurwitz formula yields:

$$2g(C) - 2 = |G^\circ| \left(2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i} \right) \right),$$

hence:

$$4 = (g(C) - 1) \left(2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i} \right) \right) \Rightarrow g(C) \leq 3 \Rightarrow g(C) = 3 \text{ and } \sum_{i=1}^r \left(1 - \frac{1}{m_i} \right) = 0.$$

Then $|G^\circ| = 2$ means $G^\circ = \mathbb{Z}_2$, $|G| = 4$ and since (10) is non-split $G = \mathbb{Z}_4$. \square

4. ISOTRIVIAL FIBRATIONS

Up to now we have considered only cases where a finite group G acts freely on a product of two curves $C \times F$, hence the quotient $(C \times F)/G$ is smooth. Now we consider cases where G does not act freely on $C \times F$. Thus the quotient $(C \times F)/G$ is a normal surface, and we study its desingularization.

Definition 4.1. Assume that $T = (C \times F)/G$, where G is a finite group of automorphisms of each factor C and F , and acts diagonally on $C \times F$. Consider the minimal resolution S of the singularities of T . The holomorphic map $f_C: S \rightarrow C' := C/G$ is called a standard isotrivial fibration if it is a relatively minimal fibration.

Remark 4.2. Let S be a minimal surface of general type with $p_g(S) = q(S) = 2$ and $S \rightarrow B$ be an isotrivial fibration with general fibre F , and $g(B) = 1$, then by [S, §1.1] S is birational to $(C \times F)/G$, $B \cong C/G$ and by (8) we have $g(C/G) = g(F/G) = 1$. Consider the minimal desingularization $\sigma: S' \rightarrow (C \times F)/G$, then the holomorphic map $f_C: S' \rightarrow C/G$ is a standard isotrivial fibration. Indeed suppose that there is a -1 -curve E in a fibre of f_C , then $\sigma(E)$ is a -1 -curve in $(C \times F)/G$, but $(C \times F)/G \rightarrow C/G \times F/G$ is a finite map and $C/G \times F/G$ is a product of two elliptic curve, and this gives a contradiction. Thus $S \rightarrow B$ is birational to a standard isotrivial fibration, and we shall deal from now on only with standard isotrivial fibrations.

By abuse of notation we shall also denote by $S \rightarrow T := (C \times F)/G$ a standard isotrivial fibration, and we shall refer to S as a standard isotrivial fibration.

Let $S \rightarrow T := (C \times F)/G$ be a standard isotrivial fibration of general type, which is not isogenous to a product of curves. To study the types of singularities of T , one looks first at the fixed points of the action of G on each curve and at the stabilizers $H \subset G$ of each point on each curve. For this part we shall mainly follow [MP].

Let C be a compact Riemann surface of genus $g \geq 2$ and let $G \leq \text{Aut}(C)$. For any $c \in G$ set $H := \langle c \rangle$ and define the set of fixed points by c as:

$$\text{Fix}_C(c) = \text{Fix}_C(H) := \{x \in C \mid cx = x\}.$$

Let us look more closely to the action of an automorphism in a neighborhood of a fixed point. Let \mathcal{D} be the unit disk and $c \in \text{Aut}(C)$ of order $m > 1$ such that $cx = x$ for $x \in C$. Then there is a unique complex primitive m -th root of unity ξ such that any lift of c to \mathcal{D} that fixes a point in \mathcal{D} is conjugate to the transformation $z \rightarrow \xi z$ in $\text{Aut}(\mathcal{D})$. We write $\xi_x(c) = \xi$ and we call ξ^{-1} the *rotation constant* of c in x . Then for each integer $q \leq m - 1$ such that $(m, q) = 1$ we define:

$$\text{Fix}_{C,q}(c) := \{x \in \text{Fix}_C(c) \mid \xi_x(c) = \xi^q\},$$

that is the set of fixed points of c with rotation constant ξ^{-q} . We have:

$$(12) \quad \text{Fix}_C(c) = \bigsqcup_{\substack{q \leq m-1 \\ (q,m)=1}} \text{Fix}_{C,q}(c).$$

Lemma 4.3. [Br, Lemma 10.4, Lemma 11.5] Assume that we are in the situation of the Riemann Existence Theorem 2.2, thus let $\mathcal{V} = (a_1, b_1, \dots, a_{g'}, b_{g'}, c_1, \dots, c_r)$ a generating vector of a finite group G of type $(g' \mid m_1, \dots, m_r)$. Let $c \in G \setminus \{1\}$ be of order m , $H = \langle c \rangle$ and $(q, m) = 1$. Then:

$$(13) \quad |\text{Fix}_C(c)| = |N_G(H)| \sum_{\substack{1 \leq i \leq r \\ m \mid m_i \\ H \sim_G \langle c_i^{m_i/m} \rangle}} \frac{1}{m_i},$$

and

$$|Fix_{C,q}(c)| = |C_G(c)| \sum_{\substack{1 \leq i \leq r \\ m|m_i \\ c \sim_G c_i^{qm_i/m}}} \frac{1}{m_i}.$$

We need the following two Corollaries.

Corollary 4.4. *Assume $c \sim_G c^q$. Then $|Fix_{C,1}(c)| = |Fix_{C,q}(c)|$.*

Corollary 4.5. *Let $c \in G$ with $|c| = 2$ and $c \in Z(G)$, then:*

$$|Fix_C(c)| = |G| \sum_{\{i|c \in \langle c_i \rangle\}} \frac{1}{m_i}.$$

Let $S \rightarrow T := (C \times F)/G$ be a standard isotrivial fibration of general type as explained in [S] paragraph (2.02) the stabilizer $H \subset G$ of a point $x \in F$ is a cyclic group (see e.g., [FK] Chap. III. 7.7), so, since the tangent representation is faithful on both factor, the only singularities that can occur on T are cyclic quotient singularities. More precisely, if H is the stabilizer of $x \in F$, then we have two cases. If H acts freely on C then T is smooth along the scheme-theoretic fibre of $\sigma: T \rightarrow F/G$ over $\bar{x} \in F/G$, and this fibre consists of the curve C/H counted with multiplicity $|H|$. Thus the smooth fibres of σ are all isomorphic to C . On the other hand if a non-trivial element of H fixes a point $y \in C$, then T has a cyclic quotient singularity in the point $(\bar{y}, \bar{x}) \in (C \times F)/G$. Let us briefly recall the definition of a cyclic quotient singularity.

Definition 4.6. *Let n and q be natural numbers with $0 < q < n$ and $(n, q) = 1$, and let ξ_n be a primitive n -th root of unity. Let the action of the cyclic group $\mathbb{Z}_n = \langle \xi_n \rangle$ on \mathbb{C}^2 be defined by $\xi_n \cdot (x, y) = (\xi_n x, \xi_n^q y)$. Then we say that the analytic space $\mathbb{C}^2/(\mathbb{Z}_n)$ has a cyclic quotient singularity of type $\frac{1}{n}(1, q)$.*

We are interested in desingularization $\sigma: S \rightarrow T$ of cyclic quotient singularities. The exceptional divisor E on the minimal resolution of such a singularity is given by a Hirzebruch-Jung string (see e.g., [R], or [BHPV]).

Definition 4.7. *A Hirzebruch-Jung string is a union $E := \cup_i^k E_i$ of smooth rational curves E_i such that:*

- $E_i^2 = -b_i \leq -2$ for all i ,
- $E_i E_j = 1$ if $|i - j| = 1$,
- $E_i E_j = 0$ if $|i - j| \geq 2$,

where the b_i 's are given by the continued fraction associated to $\frac{1}{n}(1, q)$. Indeed by the formula:

$$\frac{n}{q} = b_1 - \frac{1}{b_2 - \frac{1}{\dots - \frac{1}{b_k}}}.$$

By abuse of notation we shall refer to $[b_1, \dots, b_k]$ as the continued fraction associated to $\frac{1}{n}(1, q)$.

These observations lead to the following theorem.

Theorem 4.8. [S, Theorem 2.1] *Let $\sigma: S \rightarrow T := (C \times F)/G$ be a standard isotrivial fibration and let us consider the natural projection $\beta: S \rightarrow F/G$. Take any point over $\bar{y} \in F/G$ and let Λ denote the fibre of β over \bar{y} . Then:*

- (1) *The reduced structure of Λ is the union of an irreducible curve Y , called the central component of Λ , and either none or at least two mutually disjoint Hirzebruch-Jung strings, each meeting Y at one point. These strings are in one-to-one correspondence with the branch points of $C \rightarrow C/H$, where H is the stabilizer of \bar{y} .*
- (2) *The intersection of a string with Y is transversal, and it takes place at only one of the end components of the string.*

(3) Y is isomorphic to C/H , and has multiplicity $|H|$ in Λ .

Evidently, a similar statement holds if we consider the natural projection $\alpha: S \rightarrow C/G$.

We shall now determine the numerical invariants of isotrivial fibrations as we have done for surfaces isogenous to a product, in general the invariants will also depend on the singularities. The following Lemma, which derives from section 6 of [Ba], explains how to calculate them (for a more exhaustive treatment of cyclic singularities of isotrivial surfaces with $\chi(\mathcal{O}_S) = 1$ see also [MP]).

Lemma 4.9. *Let $\sigma: S \rightarrow T := C \times F/G$ be a standard isotrivial fibration. Then:*

$$(14) \quad K_S^2 = \frac{8(g(C) - 1)(g(F) - 1)}{|G|} + \sum_{x \in \text{Sing}(T)} h_x,$$

$$(15) \quad e(S) = \frac{4(g(C) - 1)(g(F) - 1)}{|G|} + \sum_{x \in \text{Sing}(T)} e_x,$$

where h_x and e_x depend on the type of singularity in x . If x is a cyclic quotient singularity of type $\frac{1}{n}(1, q)$ then:

$$h_x := 2 - \frac{2 + q + q'}{n} - \sum_{i=1}^k (b_i - 2),$$

$$e_x := k + 1 - \frac{1}{n},$$

where $1 \leq q' \leq n - 1$ and such that $qq' \equiv 1 \pmod{n}$, and b_i with $(1 \leq i \leq k)$ are the continued fractions data. Moreover $e_x \geq \frac{3}{2}$.

Remark 4.10. If $x \in T$ is a rational double point, i.e., x is a singularity of type $\frac{1}{n}(1, n - 1)$, we have:

$$h_x = 0, \quad e_x = \frac{(n - 1)(n + 1)}{n}.$$

In the case x is a point of type $\frac{1}{3}(1, 1)$, then the continued fraction is given by [3] hence we have:

$$h_x = -\frac{1}{3}, \quad e_x = \frac{5}{3}.$$

In the case x is a point of type $\frac{1}{4}(1, 1)$, then the continued fraction is given by [4] hence we have:

$$h_x = -1, \quad e_x = \frac{7}{4}.$$

In the case x is a point of type $\frac{1}{5}(1, 2)$, then the continued fraction is given by [3, 2] hence we have:

$$h_x = -\frac{2}{5}, \quad e_x = \frac{14}{5}.$$

Remark 4.11. Let us consider now a standard isotrivially fibred surface S with $\chi(S) = 1$, then by the Noether's formula:

$$e(S) = 12\chi(S) - K_S^2 = 12 - K_S^2.$$

Observe that together (14) and (15) yield:

$$K_S^2 = 2e(S) - \sum_{x \in \text{Sing}(T)} (2e_x - h_x),$$

Combining the above two formulas we get:

$$(16) \quad K_S^2 = 8 - \frac{1}{3} \sum_{x \in \text{Sing}(T)} (2e_x - h_x).$$

Set $B_x := 2e_x - h_x$. Let us suppose now that S is irregular, hence by Debarre's inequality we have $K_S^2 \geq 2$ (if $p_g = q = 2$ we even have $K_S^2 \geq 4$). Since S is smooth $\sum_{x \in \text{Sing}(T)} B_x \equiv 0 \pmod{3}$, and combining these two facts gives the following upper bound:

$$\sum_{x \in \text{Sing}(T)} B_x \leq 18.$$

By the Bogomolov-Miyaoka-Yau inequality $K_S^2 \leq 9$, this gives the lower bound

$$\sum_{x \in \text{Sing}(T)} B_x \geq -3.$$

Now if $\sum_{x \in \text{Sing}(T)} B_x = 0$ then T is isomorphic to S , hence it is smooth and $K_S^2 = 8$ and S is isogenous to a product. If $\sum_{x \in \text{Sing}(T)} B_x = -3$ then S is a ball quotient by the Miyaoka-Yau Theorem, which is absurd. Hence $\sum_{x \in \text{Sing}(T)} B_x \geq 0$ and

$$(17) \quad K_S^2 \leq 8.$$

We find a basket of possible singularities for T depending on K_S^2 .

Proposition 4.12. *Let $\sigma: S \rightarrow T = (C \times F)/G$ be a standard isotrivial fibration with $\chi(S) = 1$ and $p_g = q = 2$. Then the possible singularities of T are given in the following list.*

- $K_S^2 = 6$:
 - (i) $2 \times \frac{1}{2}(1, 1)$.
- $K_S^2 = 5$:
 - (i) $\frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$,
 - (ii) $2 \times \frac{1}{4}(1, 1)$,
 - (iii) $3 \times \frac{1}{2}(1, 1)$.
- $K_S^2 = 4$:
 - (i) $\frac{1}{4}(1, 1) + \frac{1}{4}(1, 3)$,
 - (ii) $2 \times \frac{1}{5}(1, 2)$,
 - (iii) $\frac{1}{2}(1, 1) + 2 \times \frac{1}{4}(1, 1)$,
 - (iv) $4 \times \frac{1}{2}(1, 1)$.

We observe that this table is just a part of a more complete table given in [MP, Proposition 4.1], where the authors give a list of all possible singularities for irregular standard isotrivial fibrations with $\chi(S) = 1$.

Remark 4.13. Recall from Remark 3.11 that the singularities of T arise from the points in $C \times F$ with non-trivial stabilizer; since the action of G on $C \times F$ is the diagonal one, it follows that $\mathcal{S}' = (\Sigma(\mathcal{V}_1) \cap \Sigma(\mathcal{V}_2)) \setminus \{1\}$ is the set of all non-trivial stabilizers for the action of G on $C \times F$. Suppose that every element of \mathcal{S}' has order 2, then we have that the singularities of T are nodes, whose number is given by:

$$(18) \quad \# \text{Nodes}(T) = \frac{2}{|G|} \sum_{c \in \mathcal{S}'} |\text{Fix}_C(c)| |\text{Fix}_F(c)|,$$

see e.g., [P1, §5].

Lemma 4.14. *Let S be as in Theorem 4.15 suppose $|\text{Sing}(T)| = 2$ or 3 and $g(F) = 2$, then the covering $C \rightarrow C/G$ has only one ramification point.*

Proof. Let us suppose that $C \rightarrow C/G$ has $r \geq 1$ ramification points. Let $i \in \{1, \dots, r\}$ and $\{m_i\}_{i=1}^r$ be the branching data. Since $|\text{Sing}(T)| = 2$ or 3 the corresponding Hirzebruch-Jung strings must belong to the same fibre of $S \rightarrow C/G$, because by theorem 4.8 each fibre must contain either none or at least two strings. It follows that, for all i except one there is a

subgroup $H \leq G$ isomorphic to \mathbb{Z}_{m_i} , which acts freely on F . Now since $g(F) = 2$ and by the Riemann-Hurwitz formula for the covering $F \rightarrow F/H$ we have:

$$1 = g(F) - 1 = m_i(g(F/H) - 1),$$

hence all the m_i 's except at most one divide 1, therefore there is only one m_i , and so only one ramification point. \square

Theorem 4.15. *Let $T := (C \times F)/G$ be a singular complex surface and S its minimal resolution of singularities such that S has $p_g = q = 2$. Then S is minimal of general type and the only possibilities for K_S^2 , $g(C)$, $g(F)$, the groups G , the branching data \mathbf{m} , the types, and the numbers of singularities of T are given in Table 2.*

Proof. **Step 1** S is minimal.

First recall from Theorem 3.6 and (8) that S is of Albanese general type and C/G and F/G are elliptic curves. If E is a (-1) -curve on S then the image of E in T is rational curve. But $T \rightarrow C/G \times F/G$ is a finite map and $C/G \times F/G$ is a product of two elliptic curve, and this gives a contradiction, hence S is minimal.

Step 2 $4 \leq K_S^2 \leq 6$.

Since S is minimal and irregular, by Debarre's inequality we have $K_S^2 \geq 2p_g = 4$, and by (17) we have $K_S^2 \leq 8$. We see from equation (16) that if $K_S^2 = 8$ then T is nonsingular, and this case cannot occur. If $K_S^2 = 7$, then by (16) we must have $\sum_{x \in \text{Sing}(T)} B_x = 3$, which means that T can have only one singularity of type $\frac{1}{2}(1, 1)$, but this contradicts Serrano's Theorem 4.8. Hence $4 \leq K_S^2 \leq 6$.

Step 3 consists of checking, once K_S^2 is fixed, if there is a standard isotrivial surface $S \rightarrow T = (C \times F)/G$ with $p_g = q = 2$ such that T has the prescribed singularities given in Proposition 4.12.

Case $K_S^2 = 6$.

In this case we have only a pair of singularities of type $\frac{1}{2}(1, 1)$, hence $K_S^2 = \frac{8(g(C)-1)(g(F)-1)}{|G|}$, and by Riemann-Hurwitz formula ($g(C/G) = 1$) we have:

$$(19) \quad g(C) - 1 = \frac{|G|}{2} \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right).$$

Combining the two formulas we have:

$$\frac{3}{2} = (g(F) - 1) \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right).$$

Suppose $r = 1$. Then $\frac{3}{2} \leq g(F) - 1 \leq 3$, hence $3 \leq g(F) \leq 4$. By symmetry we can suppose w.l.o.g. $g(C) \leq g(F)$, moreover we shall always assume this from now on.

If $g(F) = 4$ and $g(C) = 4$, then both have one branching point of order 2, and $|G| = 12$. As we have already seen A_4 is the only group of order 12 that is $(1 \mid 2)$ -generated. Choose, for example, as generating vectors for G for both coverings:

$$a_{1,1} = a_{2,1} = (123), \quad b_{1,1} = b_{2,1} = (124), \quad c_{1,1} = c_{2,1} = (12)(34).$$

We have $\mathcal{S}' = \{(12)(34), (13)(24), (14)(23)\}$. For all $c \in \mathcal{S}'$ we have:

$$|\text{Fix}_C(c)| = 2, \quad |\text{Fix}_F(c)| = 2,$$

so by equation (18) T has $\frac{2 \cdot 2 \cdot 3}{6} = 2$ nodes. Hence there exists S and this gives the last case of the Table 2.

If $g(F) = 4$ and $g(C) = 3$, then $|G| = 8$ and the branching data for F and C are respectively (4) and (2). However the commutators of D_4 and Q_8 have order 2, hence neither group is $(1 \mid 4)$ -generated, and this case is excluded.

If $g(F) = 3$, then $g(C) = 3$ and $|G| = \frac{8 \cdot 2 \cdot 2}{6} = \frac{16}{3}$ which is absurd, and this case is impossible.

Suppose $r \geq 2$. Then $g(F) - 1 \leq \frac{3}{2}$ hence $g(F) = 2$ and this is a contradiction to Lemma 4.14.

Case $K_S^2 = 5$.

We have several cases according to Proposition 4.12.

Case (i). In this case we have two singularities, one of type $\frac{1}{3}(1, 1)$ and one of type $\frac{1}{3}(1, 2)$. By Remark 4.10 we have $\sum h_x = -\frac{1}{3}$, hence we have $K_S^2 = \frac{8(g(C)-1)(g(F)-1)}{|G|} - \frac{1}{3}$, combining this formula with (19) we have:

$$\frac{4}{3} = (g(F) - 1) \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right).$$

Suppose $r = 1$, then $2 < g(F) \leq 3$.

If $g(F) = 3$ and $g(C) = 3$, then $|G| = 6$, and the branching data for both coverings are (3). As we have seen S_3 is a non-abelian group which is $(1 \mid 3)$ -generated. Choose, for example, the following generating vector for G for both coverings:

$$a_{1,1} = a_{2,1} = (12), \quad b_{1,1} = b_{2,1} = (13), \quad c_{1,1} = c_{2,1} = (123).$$

We have $\mathcal{S}' = \{(123), (132)\}$ and for all $c \in \mathcal{S}'$

$$|Fix_{C,1}(c)| = |Fix_{C,2}(c)| = 1,$$

$$|Fix_{F,1}(c)| = |Fix_{F,2}(c)| = 1.$$

So $C \times F$ contains exactly four points with non-trivial stabilizer and for each of them the G -orbit has cardinality $|G|/|(123)| = 2$. Hence T contains precisely two singular points and looking at the rotation constants we see that it has the required singularities. Hence S exists.

If $r \geq 2$ then we have only the possibility $g(F) = 2$ which is again a contradiction to Lemma 4.14.

Case(ii). In this case we have two singularities of type $\frac{1}{4}(1, 1)$. By Remark 4.10 $h_x = -1$, which yields $K_S^2 = \frac{8(g(C)-1)(g(F)-1)}{|G|} - 2$, and combining this formula with (19) we have:

$$\frac{7}{4} = (g(F) - 1) \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right).$$

Suppose $r = 1$, then $3 \leq g(F) \leq 4$.

If $g(F) = 4$ and $g(C) = 4$, then $|G| = \frac{8 \cdot 3 \cdot 3}{7}$, impossible.

If $g(F) = 4$ then $g(C) = 3$ and $|G| = \frac{8 \cdot 3 \cdot 2}{7}$, impossible.

If $g(F) = 3$ and $g(C) = 3$, then $|G| = \frac{8 \cdot 2 \cdot 2}{7}$, impossible.

Suppose $r \geq 2$ then the only possibility is $g(F) = 2$ which is again a contradiction to Lemma 4.14.

Case(iii). In this case we have three singularities of type $\frac{1}{2}(1, 1)$. By Remark 4.10 we have $\sum h_x = 0$, which yields $K_S^2 = \frac{8(g(C)-1)(g(F)-1)}{|G|}$, and combining this formula with (19) we obtain:

$$\frac{5}{4} = (g(F) - 1) \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right).$$

Suppose $r = 1$, then $2 < g(F) \leq 3$.

If $g(F) = 3$ and $g(C) = 3$, then $|G| = \frac{8 \cdot 2 \cdot 2}{5}$, impossible.

Suppose $r \geq 2$ then the only possibility is $g(F) = 2$ which is again a contradiction to Lemma 4.14.

Case $K_S^2 = 4$.

We have several cases according to Proposition 4.12.

Case (i). In this case we have two singularities one of type $\frac{1}{4}(1, 1)$ and one of type $\frac{1}{4}(1, 3)$. By Remark 4.10 we have $\sum h_x = -1$, hence we have $K_S^2 = \frac{8(g(C)-1)(g(F)-1)}{|G|} - 1$. Combining

this formula with (19) we obtain:

$$\frac{5}{4} = (g(F) - 1) \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right).$$

Suppose $r = 1$, then $2 < g(F) \leq 3$.

If $g(F) = 3$ and $g(C) = 3$, then $|G| = \frac{8 \cdot 2 \cdot 2}{5}$, impossible.

If $r \geq 2$ then we have only the possibility $g(F) = 2$ which is again a contradiction to Lemma 4.14.

Case(ii). In this case we have two singularities of type $\frac{1}{5}(1, 2)$. By Remark 4.10 we have $\sum h_x = -\frac{4}{5}$, which yields $K_S^2 = \frac{8(g(C)-1)(g(F)-1)}{|G|} - \frac{4}{5}$, combining this formula with (19) we have:

$$\frac{6}{5} = (g(F) - 1) \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right).$$

Suppose $r = 1$, then $2 < g(F) \leq 3$.

If $g(F) = 3$ and $g(C) = 3$, then $|G| = \frac{8 \cdot 2 \cdot 2 \cdot 5}{24}$, impossible.

Suppose $r \geq 2$, then the only possibility is $g(F) = 2$ which is again a contradiction to Lemma 4.14.

Case(iii). In this case we have three singularities one of type $\frac{1}{2}(1, 1)$ and two of type $\frac{1}{4}(1, 1)$. By Remark 4.10 we have $\sum h_x = -2$, which yields $K_S^2 = \frac{8(g(C)-1)(g(F)-1)}{|G|} - 2$, combining this formula with (19) we have:

$$\frac{6}{4} = (g(F) - 1) \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right).$$

Suppose $r = 1$, then $3 \leq g(F) \leq 4$.

If $g(F) = 4$ and $g(C) = 4$, then $|G| = 12$, and the branching data are (2) for both coverings, but this contradicts the fact that we have singularities of type $\frac{1}{4}(1, 1)$, hence this case is impossible.

If $g(F) = 4$ and $g(C) = 3$, then $|G| = 8$ and the branching data for F and C are respectively (4), (2). We have already seen that there is no non-abelian group of order 8 which is $(1 \mid 4)$ -generated, hence the case is excluded.

If $g(F) = 3$, then $g(C) = 3$ and $|G| = \frac{8 \cdot 2 \cdot 2}{6}$, impossible.

Suppose $r \geq 2$ then the only possibility is $g(F) = 2$ which is again a contradiction to Lemma 4.14.

Case(iv). In this case we have four singularities of type $\frac{1}{2}(1, 1)$. By Remark 4.10 we have $\sum h_x = 0$, which yields $K_S^2 = \frac{8(g(C)-1)(g(F)-1)}{|G|}$, and combining this formula with (19) we have:

$$1 = (g(F) - 1) \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right).$$

Suppose $r = 1$, then $2 < g(F) \leq 3$.

If $g(F) = 3$ and $g(C) = 3$, then $|G| = 8$, and the branching data are (2) for both coverings. The groups D_4 and Q_8 are $(1 \mid 2)$ -generated, choose, for example, as generating vector for Q_8 for both coverings:

$$a_{1,1} = a_{2,1} = i, \quad b_{1,1} = b_{2,1} = j, \quad c_{1,1} = c_{2,1} = -1,$$

and for D_4 for both coverings:

$$a_{1,1} = a_{2,1} = x, \quad b_{1,1} = b_{2,1} = y, \quad c_{1,1} = c_{2,1} = x^2.$$

We have in both cases $\mathcal{S}' = Z(G) \setminus \{1\}$, and by Corollary 4.5 for $c \in \mathcal{S}'$

$$|Fix_C(c)| = |Fix_F(c)| = 4.$$

Then by equation (18) T has exactly $\frac{2 \cdot 4 \cdot 4}{8} = 4$ nodes in both cases. Hence S exists.

Suppose $r \geq 2$ then the only possibility is $g(F) = 2$ and $r = 2$. In this case there are more than three singularities so Lemma 4.14 does not apply, hence $g(C) = 2$, $|G| = 2$ and both coverings

have branching data $(2, 2)$. Let x be the generator of G , we have $\mathcal{S}' = \{x\}$, and Corollary 4.5 implies

$$|Fix_C(x)| = |Fix_F(x)| = 2.$$

Then by equation (18) T has exactly $\frac{2 \cdot 2 \cdot 2}{2} = 4$ nodes. This yields the first case in the table. \square

We notice that the first case in Table 2 was already given in [Z].

5. MODULI SPACES

By a celebrated Theorem of Gieseker (see [Gie]), once the two invariants of a minimal surface S of general type, K_S^2 and $\chi(S)$, are fixed, then there exists a quasiprojective moduli space $\mathcal{M}_{K_S^2, \chi(S)}$ of minimal smooth complex surfaces of general type with those invariants, and this space consists of a finite number of connected components. The union \mathcal{M} over all admissible pairs of invariants (K^2, χ) of these spaces is called the *moduli space of surfaces of general type*.

In [C1], Catanese studied the moduli space of surfaces isogenous to a product of curves (see Theorem 4.14). As a result, one obtains that the moduli space of surfaces isogenous to a product of curves $\mathcal{M}_{(G, (\tau_1, \tau_2))}$ with fixed invariants: a finite group G and types (τ_1, τ_2) (where the types $\tau_i := (g'_i | m_{i,1}, \dots, m_{i,r_i})$, for $i = 1, 2$, are defined in 2.1) for the unmixed case (while only G and one type τ in the mixed case), consists of a finite number of irreducible connected components of \mathcal{M} .

The surfaces we are studying are quotients of products of curves and to study their moduli space one has to look first at the moduli space of Riemann surfaces.

Let $\mathcal{M}_{g,r}$ denote the *moduli space of Riemann surfaces of genus g with r ordered marked points*. By permuting the marked points on the Riemann surfaces, the permutation group S_r acts naturally on this space; the moduli space $\mathcal{M}_{g,[r]} = \mathcal{M}_{g,r}/S_r$ classifies the Riemann surfaces of genus g with r unordered marked points. By Teichmüller theory these spaces are quotient of a contractible spaces $\mathcal{T}_{g,r}$ of complex dimension $3g - 3 + r$, if $g' = 0$ and $r \geq 3$, or $g' = 1$ and $r \geq 1$ or $g' \geq 2$, called the *Teichmüller spaces*, by the action of discrete groups called the *full mapping class groups* $Map_{g,[r]}$.

In [BC, Theorem 1.3] is given a method to calculate the number of connected components of the moduli spaces $\mathcal{M}_{(G, (\tau_1, \tau_2))}$ of surfaces isogenous to a product of unmixed type using Teichmüller theory, while in [BCG, Proposition 5.5] is treated the mixed case.

Notice that the dimension of the space $\mathcal{M}_{(G, (\tau_1, \tau_2))}$, with types $\tau_i := (g'_i | m_{i,1}, \dots, m_{i,r_i})$ for $i = 1, 2$, is precisely $\dim \mathcal{M}_{(G, (\tau_1, \tau_2))} = 3g'_1 - 3 + r_1 + 3g'_2 - 3 + r_2$, while in the mixed case, if $\tau = (g' | m_1, \dots, m_r)$, then $\dim \mathcal{M}_{(G, \tau)} = 3g' - 3 + r$. This is enough to determine the numbers in the column \dim of Table 1.

Definition 5.1. *Let M be a differentiable manifold, then the mapping class group (or Dehn group) of M is the group:*

$$\text{Map}(M) := \pi_0(\text{Diff}^+(M)) = \text{Diff}^+(M)/\text{Diff}^0(M),$$

where $\text{Diff}^+(M)$ is the group of orientation preserving diffeomorphisms of M and $\text{Diff}^0(M)$ is the subgroup of diffeomorphisms of M isotopic to the identity.

If M is a compact complex curve of genus g' we will use the following notations:

- (1) We denote the mapping class group of M by $\text{Map}_{g'}$.
- (2) If we consider r points p_1, \dots, p_r on M we define:

$$\text{Map}_{g',[r]} := \pi_0(\text{Diff}^+(M - \{p_1, \dots, p_r\})),$$

and this is known as the full mapping class group.

There is an easy way to present the mapping class group of a curve using half twists and Dehn twists see e.g., [C2, D].

Theorem 5.2. *The group Map_2 is generated by the Dehn twists with respect to the five curves in the figure:*

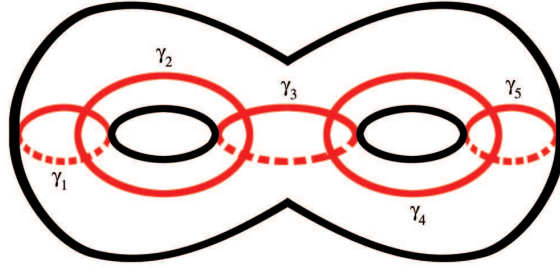


Figure 1.

The corresponding relations are the following:

- (1) $t_{\gamma_i} t_{\gamma_j} = t_{\gamma_j} t_{\gamma_i}$ if $|i - j| \geq 2$, $1 \leq i, j \leq 5$,
- (2) $t_{\gamma_i} t_{\gamma_{i+1}} t_{\gamma_i} = t_{\gamma_{i+1}} t_{\gamma_i} t_{\gamma_{i+1}}$, $1 \leq i \leq 4$,
- (3) $(t_{\gamma_1} t_{\gamma_2} t_{\gamma_3} t_{\gamma_4} t_{\gamma_5})^6 = 1$,
- (4) $(t_{\gamma_1} t_{\gamma_2} t_{\gamma_3} t_{\gamma_4} t_{\gamma_5}^2 t_{\gamma_4} t_{\gamma_3} t_{\gamma_2} t_{\gamma_1})^2 = 1$,
- (5) $[t_{\gamma_1} t_{\gamma_2} t_{\gamma_3} t_{\gamma_4} t_{\gamma_5}^2 t_{\gamma_4} t_{\gamma_3} t_{\gamma_2} t_{\gamma_1}, t_{\gamma_i}] = 1$, $1 \leq i \leq 5$.

A proof of the above theorem can be found in [Bir, Theorem 4.8].

Theorem 5.3. Let $\Gamma = \Gamma(g' \mid m_1, \dots, m_r)$ be an orbifold surface group and $\text{Out}^+(\Gamma)$ the group of orientation preseving outer automorphisms of Γ . Then there is an isomorphism of groups:

$$\text{Out}^+(\Gamma) \cong \text{Map}_{g', [r]}.$$

This is a classical result cf. e.g., [Sch, Theorem 2.2.1] and [Macl, §4].

Moreover let G be a finite group $(g' \mid m_1, \dots, m_r)$ -generated, then the action of $\text{Out}^+(\Gamma) \cong \text{Map}_{g', [r]}$ on the generators of $\Gamma(g' \mid m_1, \dots, m_r)$ induces an action on the generating vectors of G via composition with admissible epimorphisms.

Definition 5.4. Let G be a finite group $(g' \mid m_1, \dots, m_r)$ -generated. If two generating vectors \mathcal{V}_1 and \mathcal{V}_2 are in the same $\text{Map}_{g', [r]}$ -orbit, we say that they are related by a Hurwitz move (or are Hurwitz equivalent).

Now we calculate the Hurwitz moves on the generating vectors of type $(2 \mid -)$.

Proposition 5.5. Up to inner automorphism, the action of Map_2 on $\Gamma(2 \mid -)$ is given by:

$$\begin{aligned}
 t_{\gamma_2} : \begin{cases} \alpha_1 & \rightarrow \alpha_1 \\ \beta_1 & \rightarrow \beta_1 \alpha_1 \\ \alpha_2 & \rightarrow \alpha_2 \\ \beta_2 & \rightarrow \beta_2 \end{cases} & \quad t_{\gamma_1} : \begin{cases} \alpha_1 & \rightarrow \alpha_1 \beta_1^{-1} \\ \beta_1 & \rightarrow \beta_1 \\ \alpha_2 & \rightarrow \alpha_2 \\ \beta_2 & \rightarrow \beta_2 \end{cases} \\
 t_{\gamma_5} : \begin{cases} \alpha_1 & \rightarrow \alpha_1 \\ \beta_1 & \rightarrow \beta_1 \\ \alpha_2 & \rightarrow \alpha_2 \beta_2^{-1} \\ \beta_2 & \rightarrow \beta_2 \end{cases} & \quad t_{\gamma_4} : \begin{cases} \alpha_1 & \rightarrow \alpha_1 \\ \beta_1 & \rightarrow \beta_1 \\ \alpha_2 & \rightarrow \alpha_2 \\ \beta_2 & \rightarrow \beta_2 \alpha_2 \end{cases} \\
 t_{\gamma_3} : \begin{cases} \alpha_1 & \rightarrow \alpha_1 x^{-1} \\ \beta_1 & \rightarrow x \beta_1 x^{-1} \\ \alpha_2 & \rightarrow x \alpha_2 \\ \beta_2 & \rightarrow \beta_2. \end{cases}
 \end{aligned}$$

where $\alpha_1, \alpha_2, \beta_1$ and β_2 are the generators of $\Gamma(2 \mid -)$ and $x = \beta_2^{-1} \alpha_1 \beta_1 \alpha_1^{-1} = \alpha_2 \beta_2^{-1} \alpha_2^{-1} \beta_1$.

Proof. One notices that a Riemann surface of genus 2 is a connected sum of two tori. Then one can use the results given in [P1, Proposition 1.10] to calculate the Dehn twists about the curves $\gamma_1, \gamma_2, \gamma_4, \gamma_5$ of Figure 1, considering the action on the two different tori. This gives the

actions t_{γ_1} , t_{γ_2} , t_{γ_4} and t_{γ_5} .

Then the only Dehn twist left to calculate is the one with respect to the curve γ_3 as in Figure 2.

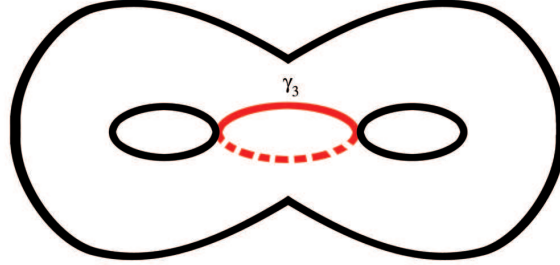


Figure 2.

Choose the generators of the fundamental group as in Figure 3:

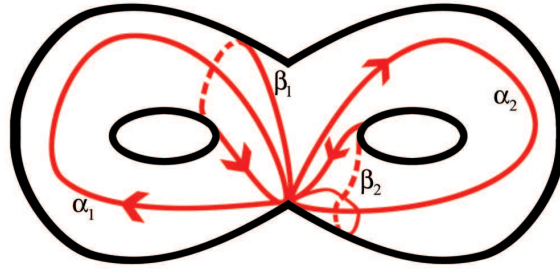


Figure 3.

One sees that the only curves which have to be twisted are α_1 , β_1 and α_2 because the other is disjoint from γ_3 . In Figure 4 one sees the Dehn twist of α_1 with respect to γ_3 . Following the curve one constructs the image of α_1 under the map t_{γ_3} .

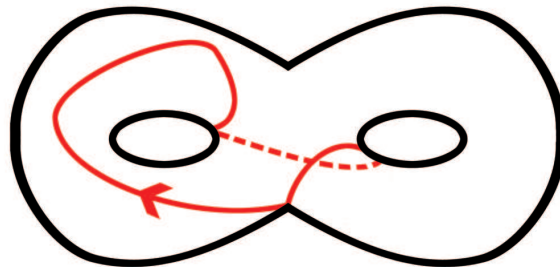


Figure 4.

In Figure 5 we give the Dehn twist of β_1 with respect to γ_3 .

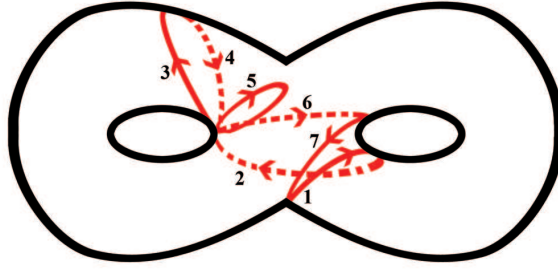


Figure 5.

In the last Figure we give the Dehn twist of α_2 with respect to γ_3 which completes the proof.

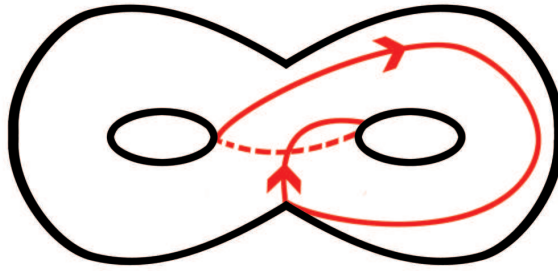


Figure 6.

□

Corollary 5.6. *Let G be a finite group and let $\mathcal{V} = (a_1, b_1, a_2, b_2)$ be a generating vector for G of type $\tau = (2 \mid -)$. Then the Hurwitz moves on the set of generating vectors of G of type τ are generated by:*

$$\begin{aligned}
 \mathbf{1} : & \begin{cases} a_1 \rightarrow a_1 \\ b_1 \rightarrow b_1 a_1 \\ a_2 \rightarrow a_2 \\ b_2 \rightarrow b_2 \end{cases} & \mathbf{2} : & \begin{cases} a_1 \rightarrow a_1 b_1^{-1} \\ b_1 \rightarrow b_1 \\ a_2 \rightarrow a_2 \\ b_2 \rightarrow b_2 \end{cases} \\
 \mathbf{3} : & \begin{cases} a_1 \rightarrow a_1 \\ b_1 \rightarrow b_1 \\ a_2 \rightarrow a_2 b_2^{-1} \\ b_2 \rightarrow b_2 \end{cases} & \mathbf{4} : & \begin{cases} a_1 \rightarrow a_1 \\ b_1 \rightarrow b_1 \\ a_2 \rightarrow a_2 \\ b_2 \rightarrow b_2 a_2 \end{cases} \\
 \mathbf{5} : & \begin{cases} a_1 \rightarrow a_1 x^{-1} \\ b_1 \rightarrow x b_1 x^{-1} \\ a_2 \rightarrow x a_2 \\ b_2 \rightarrow b_2. \end{cases}
 \end{aligned}$$

where $x = b_2^{-1} a_1 b_1 a_1^{-1} = a_2 b_2^{-1} a_2^{-1} b_1$.

Proof. This follows directly from Proposition 5.5. □

For the Hurwitz moves for types $(0 \mid m_1, \dots, m_r)$, $(1 \mid 1)$ and $(1 \mid 2)$ we refer to [P1] for proofs. If $\mathcal{V} := \{c_1, \dots, c_r\}$ is an unordered generating vector of type $\tau := (0 \mid m_1, \dots, m_r)$ then the Hurwitz moves on the set of unordered generating vectors of G of type τ are generated, for $1 \leq i \leq r-1$, by

$$\mathbf{i} : \begin{cases} c_i & \longrightarrow c_{i+1} \\ c_{i+1} & \longrightarrow c_{i+1}^{-1} c_i c_{i+1} \\ c_j & \longrightarrow c_j \text{ if } j \neq i, i+1. \end{cases}$$

If $\mathcal{V} := \{a_1, b_1, c_1\}$ is of type $(1 \mid m)$ then the Hurwitz moves are generated by

$$\mathbf{1}: \begin{cases} c_1 \longrightarrow c_1 \\ a_1 \longrightarrow a_1 \\ b_1 \longrightarrow b_1 a_1 \end{cases} \quad \mathbf{2}: \begin{cases} c_1 \longrightarrow c_1 \\ a_1 \longrightarrow a_1 b_1^{-1} \\ b_1 \longrightarrow b_1. \end{cases}$$

If $\mathcal{V} := \{a_1, b_1, c_1, c_2\}$ is of type $(1 \mid m^2)$ then the Hurwitz moves are generated by

$$\begin{aligned} \mathbf{1}: & \begin{cases} c_1 \longrightarrow c_1 \\ c_2 \longrightarrow c_2 \\ a_1 \longrightarrow a_1 \\ b_1 \longrightarrow b_1 a_1 \end{cases} & \mathbf{2}: & \begin{cases} c_1 \longrightarrow c_1 \\ c_2 \longrightarrow c_2 \\ a_1 \longrightarrow a_1 b_1^{-1} \\ b_1 \longrightarrow b_1 \end{cases} \\ \mathbf{3}: & \begin{cases} c_1 \longrightarrow c_1 \\ c_2 \longrightarrow a_1 b_1^{-1} a_1^{-1} c_2 a_1 b_1 a_1^{-1} \\ a_1 \longrightarrow b_1^{-1} c_1 a_1 \\ b_1 \longrightarrow b_1 \end{cases} & \mathbf{4}: & \begin{cases} c_1 \longrightarrow b_1^{-1} a_1^{-1} c_2 a_1 b_1 \\ c_2 \longrightarrow a_1^{-1} b_1^{-1} c_1 b_1 a_1 \\ a_1 \longrightarrow a_1^{-1} \\ b_1 \longrightarrow b_1^{-1}. \end{cases} \end{aligned}$$

The following theorem is a natural generalization of [BC, Theorem 1.3].

Theorem 5.7. *Let S be a surface isogenous to a product of unmixed type with $p_g = q = 2$. Then to S we attach its finite group G (up to isomorphism) and the equivalent class of a pair of disjoint unordered generating vectors $(\mathcal{V}_1, \mathcal{V}_2)$ of type $((2 \mid -), (0 \mid \mathbf{m}))$ (or $((1 \mid \mathbf{n}_1), (1 \mid \mathbf{n}_2))$) of G , under the equivalent relation generated by:*

- (1) *Hurwitz equivalence and $\text{Inn}(G)$ on \mathcal{V}_1 ,*
- (2) *Hurwitz equivalence and $\text{Inn}(G)$ on \mathcal{V}_2 ,*
- (3) *simultaneous conjugation of \mathcal{V}_1 and \mathcal{V}_2 by an element $\lambda \in \text{Aut}(G)$, i.e., we let $(\mathcal{V}_1, \mathcal{V}_2)$ be equivalent to $(\lambda(\mathcal{V}_1), \lambda(\mathcal{V}_2))$.*

Then two surfaces S and S' are deformation equivalent if and only if the corresponding pairs of generating vectors are in the same equivalence class.

Proof. We can use the same argument of [BC, Theorem 1.3], one has only to substitute the braid actions with the action of the appropriate mapping class group. \square

To calculate the number of connected components is then the same as to calculate the number of all possible generating pairs modulo the equivalence relation defined above. Since this task may be too hard to be achieved by hand with S. Rollenske we developed a program in GAP4 which calculates the number of these pairs. The appendix by Rollenske is devoted to explain how the program works, for the script of the program see [Pe]. The result of this computation is the last column of Table 1.

Remark 5.8. Notice that in the program is not implemented the action generated by the mapping class group and the group of inner automorphisms of the group G , but only the action of the mapping class group, hence we do not act with full group prescribed by Theorem 5.7.

However this does not effect the result above. Indeed in the cases where there is only one orbit this is not a problem. It is not a problem neither if there are two orbits and the group G is abelian, because in this case the inner automorphisms act trivially.

But there are three cases where the group G is not abelian and we have two orbits. In all three cases the two pairs of generating vectors are of the form $(\mathcal{V}_1, \mathcal{V}_2)$ and $(\mathcal{V}_1, \mathcal{W}_2)$. Then we used the program to calculate the orbits, only on the right side of the pairs, under the action of the group generated by the mapping class group and the group of automorphisms of G . Notice that this group contains the group generated by the mapping class group and the group of inner automorphisms. As result we have two orbits, hence we have two orbits also for the action of the group prescribed in Theorem 5.7.

Moreover in [Pe] are exhibited the pairs $(\mathcal{V}_1, \mathcal{V}_2)$ of generating vectors which give the surfaces isogenous to a product of curves of unmixed type with $p_g = q = 2$ given in Table 1.

For the mixed case we notice that there is only one connected component of dimension 3 of the moduli space corresponding to the item labelled by Mix in Table 1. This comes directly from the proof of Theorem 3.15 and from [BCG, Proposition 5.5] adapted to this case. Indeed let us denote by $\mathcal{M}_{(G,\tau)}$ the moduli space of isomorphism classes of surfaces isogenous to a product of curves of mixed type admitting the data (G, τ) , where $\tau = (g' | m_1, \dots, m_r)$. The number of connected components is equal to the number of classes of generating vectors of type τ of G° modulo the action given by $\text{Map}_{g',[r]} \times \text{Aut}(G)$ where the first group acts via Hurwitz moves. In our case set $\mathbb{Z}_4 = \langle x \mid x^4 = 1 \rangle$, and since $G^\circ = \mathbb{Z}_2$, then the only generating vector of type $(2 \mid -)$ is given by:

$$a_1 = x^2, \quad b_1 = 1, \quad a_2 = 1, \quad b_2 = 1.$$

This computation and the following proposition conclude the proof of Theorem 1.1.

Proposition 5.9. *Each item in Table 2 provides exactly one irreducible subvariety of the moduli space of surfaces of general type.*

Proof. Recall from Theorem 4.15 that each item in Table 2 gives rise to a surface S of general type which is the minimal desingularization of $T = (C \times F)/G$ and both C/G and F/G are elliptic curves.

To see that each item in the Table 2 gives rise only to one topological type, we proceed analyzing case by case. We have to prove that each pair of generating vectors is unique up to Hurwitz moves and simultaneous conjugation. Hence for the first case there is nothing to prove.

For the other cases the groups G are all $(1 \mid m_1)$ -generated, and denote by a_1, b_1 and c_1 the elements of a generating vector ($|c_1| = m_1$). Recall that the Hurwitz moves in this case are generated by (see [P1, Corollary 1.11]):

$$\mathbf{1} : \begin{cases} a_1 \rightarrow a_1 \\ b_1 \rightarrow b_1 a_1 \\ c_1 \rightarrow c_1 \end{cases} \quad \mathbf{2} : \begin{cases} a_1 \rightarrow a_1 b_1^{-1} \\ b_1 \rightarrow b_1 \\ c_1 \rightarrow c_1. \end{cases}$$

Notice that in all the cases the groups G have the property that $[G, G] - \{id\}$ consists of a unique conjugacy class. Hence we can fix $c_1 \in [G, G]$.

In case $G = D_4$, let us fix a rotation x and a reflection y . For what we said $c_1 = x^2$. Moreover we see that a_1 and b_1 cannot be both rotations, and up to Hurwitz moves we can assume that are both reflections. The two reflections must also be in two different conjugacy classes in order to generate G . Applying then simultaneous conjugation we see that the generating vector is unique.

In case $G = Q_8$, $c_1 = -1$ and since the vector must generate G , up to simultaneous conjugation the pair (a_1, b_1) is one of the following: $(i, j), (j, i), (i, k), (k, i), (k, j), (j, k)$. By Hurwitz moves all the pairs are equivalent to the one: $a_1 = i$ and $b_1 = j$, hence the generating vector is unique.

In case $G = S_3$, $c_1 = (123)$, and since the vector must generate G , a_1 and b_1 cannot be both 3-cycles. Moreover up to Hurwitz moves we can assume that both a_1 and b_1 are transpositions. Since all the transpositions in S_3 are conjugate, we see that the generating vector is unique.

In the last case $G = A_4$, $c_1 = (12)(34)$. To generate G we need a 3-cycle, hence a_1 and b_1 cannot be both 2-2-cycles. Up to Hurwitz moves we can suppose that both are 3-cycles, which however might be in different conjugacy classes. But again applying Hurwitz moves we can suppose that they are in the same conjugacy class, hence the generating vector is unique.

In the end by Theorem 5.4 [C1] we have that isotrivial fibred surfaces with fixed topological type form a union of irreducible subvarieties of the moduli space of surfaces of general type. Here for each case we have only one irreducible variety, whose dimension is 2 for all the cases except the first where the dimension of the variety is 4. The calculation of the dimension is done in the same way as for surfaces isogenous to a product. \square

Since our program is written in all generality, we can complete the classification of the surfaces isogenous to a product with $p_g = q = 1$ given in [CP] adding the dimension of the moduli spaces and the number of connected components.

Theorem 5.10 ([CP]). *Let $S = (C \times F)/G$ be a surface with $p_g = q = 1$ isogenous to a product of curve. Then S is minimal of general type and the occurrence for $g(C)$, $g(F)$ and G are precisely those in the Table 3 in the Appendix, where \dim is the dimension of the moduli space, and n is the number of connected components.*

6. FUNDAMENTAL GROUPS

To study the component of the moduli space relative to a surface S it is sometimes useful to know the fundamental group of the surface in question. In this section we will calculate the fundamental group of the found isotrivial surfaces. In case of surface isogenous to a product of curves we have the following.

Proposition 6.1 ([C1]). *Let $S := (C_1 \times C_2)/G$ be isogenous to a product of curves. Then the fundamental group of S sits in an exact sequence:*

$$1 \longrightarrow \Pi_{g_1} \times \Pi_{g_2} \longrightarrow \pi_1(S) \longrightarrow G \longrightarrow 1,$$

where $\Pi_{g_i} := \pi_1(C_i)$, and this extension is determined by the associated maps $G \rightarrow \text{Map}_{g_i}$ to the respective Teichmüller modular groups.

By [BCGP] there is a similar description of the fundamental group of isotrivial surfaces too, which enables us to describe the fundamental group of the surfaces of Table 2. Following [BCGP] we have:

Definition 6.2. *Let G be a finite group and let $\theta_i : \Gamma_i := \Gamma(g'_i | m_1, \dots, m_{k_i}) \rightarrow G$ be two admissible epimorphisms. We define the fibre product $\mathbb{H} := \mathbb{H}(G, \theta_1, \theta_2)$ by:*

$$\mathbb{H} := \{(x, y) \in \Gamma_1 \times \Gamma_2 \mid \theta_1(x) = \theta_2(y)\}.$$

In fact \mathbb{H} is defined by the cartesian diagram:

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\quad} & \Gamma_1 \\ \downarrow & & \downarrow \theta_1 \\ \Gamma_2 & \xrightarrow{\theta_2} & G. \end{array}$$

Definition 6.3. *Let H be a group. Then its torsion subgroup $\mathbf{Tors}(H)$ is the (normal) subgroup generated by elements of finite order in H .*

Proposition 6.4 (Proposition 3.4 [BCGP]). *Let C_1, C_2 be compact Riemann surfaces of respective genera $g_i \geq 2$ and let G be a finite group acting faithfully on each C_i as a group of biholomorphic transformations. Let $T = (C_1 \times C_2)/G$, and denote by S a minimal desingularization of T . Then the fundamental group $\pi_1(T) \cong \pi_1(S)$. Moreover let $\theta_i : \Gamma_i := \Gamma(g'_i | m_1, \dots, m_{k_i}) \rightarrow G$ be two admissible epimorphisms. Then $\pi_1(T) \cong \mathbb{H}/\mathbf{Tors}(\mathbb{H})$.*

Moreover we have the following structure theorem in the hypothesis of the above theorem.

Theorem 6.5 (Theorem 0.3 [BCGP]). *The fundamental group $\pi_1(S)$ has a normal subgroup \mathcal{N} of finite index which is isomorphic to the product of surface groups, i.e., there are natural numbers $h_1, h_2 \geq 0$ such that $\mathcal{N} \cong \pi_1(\hat{C}_1) \times \pi_1(\hat{C}_2)$, where \hat{C}_i are smooth curves of genus h_i respectively.*

We have then the following theorem.

Theorem 6.6. *The fundamental group of the surfaces given by the first 4 items in Table 2 is \mathbb{Z}^4 . While the fundamental group P of the last surface fits into the exact sequence:*

$$1 \longrightarrow \mathbb{Z}^2 \times \mathbb{Z}^2 \longrightarrow P \longrightarrow D_4 \vee D_4 \longrightarrow 1,$$

where $D_4 \vee D_4$ is the central product of D_4 times D_4 , which is an extraspecial group of order 32.

Proof. To compute a presentation of the fundamental group we use the program implemented in MAGMA given in [BCGP], with few modifications for orbifold groups $\Gamma(g|m_1, \dots, m_k)$ with $g = 1$. The program gives us a presentation for all the groups. In the first 4 cases of Table 2, one has \mathbb{Z}^4 . The last case we have a presentation given by:

Finitely presented group P on 4 generators Relations

$$\begin{aligned} [P_1, P_2] &= Id(P) \\ [P_3, P_2] &= Id(P) \\ [P_4, P_3] &= Id(P) \\ P_2^{-1} * P_1^{-1} * P_4^{-1} * P_2 * P_1 * P_4 &= Id(P) \\ P_1 * P_3^{-1} * P_4 * P_1^{-1} * P_4^{-1} * P_3 &= Id(P) \\ P_1^{-2} * P_4 * P_1^2 * P_4^{-1} &= Id(P) \\ P_3^{-1} * P_1 * P_3 * P_4 * P_1^{-1} * P_4^{-1} &= Id(P) \\ P_2^{-1} * P_4^2 * P_2 * P_4^{-2} &= Id(P) \end{aligned}$$

Form this presentation one notices that the square of the generators all lie in the center of P . The core C of the subgroup $\langle P_1, P_2, P_3^2, P_4^2 \rangle$ is isomorphic to \mathbb{Z}^4 and the group P/C is identify by MAGMA as the small group of order 32 and MAGMA library-number 49, which is the central product of D_4 times D_4 , known as the extraspecial group $D_4 \curlyvee D_4$. After inspection one sees that 32 is the minimal index. \square

7. APPENDIX: THE COMPUTATION OF THE NUMBER OF CONNECTED COMPONENTS (SÖNKE ROLLENKE)

In this appendix we want to explain the strategies used to compute the number of connected components of the Moduli space of surfaces isogenous to a product with fixed invariants. All computations have been performed using the computer algebra system GAP [GAP].

For the following discussion let us fix a group G together with two types of generating vectors $\tau_i := (g_i \mid m_1, \dots, m_{k_i})$. We want to find the number μ of components of the moduli space of surfaces isogenous to a product which can be constructed from this data.

At first sight it seems a pretty simple thing to do since we have the group-theoretical description from the result of Bauer and Catanese (see Theorem 5.7) and could try the following naïve approach:

- Calculate all possible generating vectors for G of the given types.
- Check the compatibility of these pairs, i.e., if the action of G on $C \times F$ is free, and form the set of all such possible pairs.
- Calculate the orbits under the group action given in Theorem 5.7.

Then every orbit corresponds to exactly one component of the moduli space.

But this simple approach turns out not to be feasible for the following reason: the orbits are simply too big to be calculated in a reasonable amount of time (and memory). To give a concrete example we can consider $G = GL_2(\mathbb{F}_3)$, which has number (48, 29) in GAP, $\tau_1 = (2 \mid -)$ and $\tau_2 = (0 \mid 2, 3, 8)$. Then there are 59719680 pairs of generating vectors which give rise to a surface isogenous to a product with $p_g = q = 2$ but there is only 1 orbit, i.e., 1 component of the moduli space.

Therefore we need a refined strategy in order to avoid the calculation of the complete orbits. To explain it we need some notation: let \mathfrak{V}_i be the set of generating vectors of type τ_i for G and let $\mathfrak{X} \subset \mathfrak{V}_1 \times \mathfrak{V}_2$ be the set of all compatible pairs of generating vectors.

Let H be the subgroup of the group of permutations of $\mathfrak{V}_1 \times \mathfrak{V}_2$ generated by the action of the mapping class groups on both factors and the diagonal action of the automorphism group of G as described in Theorem 5.7. We denote by H_i the restriction of the action of H to the component \mathfrak{V}_i .

The action of H restricts to \mathfrak{X} and μ coincides with number of orbits in \mathfrak{X} .

The following, mostly obvious, lemma allows us to greatly simplify the calculations.

Lemma 7.1. *Let M_i be the mapping class group acting on \mathfrak{V}_i and let $(\mathcal{V}_1, \mathcal{V}_2)$ and $(\mathcal{W}_1, \mathcal{W}_2)$ be two pairs of generating vectors. Then*

- (1) *If \mathcal{V}_1 and \mathcal{W}_1 lie in the same M_1 -orbit and \mathcal{V}_2 and \mathcal{W}_2 lie in the same M_2 -orbit then $(\mathcal{V}_1, \mathcal{V}_2)$ and $(\mathcal{W}_1, \mathcal{W}_2)$ lie in the same H -orbit.*
- (2) *If \mathcal{V}_1 and \mathcal{W}_1 do not lie in the same H_1 -orbit then $(\mathcal{V}_1, \mathcal{V}_2)$ and $(\mathcal{W}_1, \mathcal{W}_2)$ lie in different H -orbits.*

Thus our revised algorithm takes roughly the following form:

- Calculate a set \mathfrak{R}_1 of representants of the H_1 -orbits on \mathfrak{V}_1 , the generating vectors of type τ_1 , and a set \mathfrak{R}_2 of representants of M_2 -orbits on \mathfrak{V}_2 .
- After testing the pairs in $\mathfrak{R}_1 \times \mathfrak{R}_2$ for compatibility we obtain a set of pairs $\mathfrak{R} \subset \mathfrak{X}$. Each orbit in \mathfrak{X} contains at least 1 element in \mathfrak{R} by 7.1 (i).
- We already have some lower bound on the number of components: if $(\mathcal{V}_1, \mathcal{V}_2), (\mathcal{W}_1, \mathcal{W}_2) \in \mathfrak{R}$ then, by 7.1 (ii), they lie in different orbits if $\mathcal{V}_1 \neq \mathcal{W}_1$ or if \mathcal{V}_2 and \mathcal{W}_2 lie in different H_2 -orbits.
- It remains to calculate the full orbit only in the following case: there are $(\mathcal{V}_1, \mathcal{V}_2), (\mathcal{W}_1, \mathcal{W}_2) \in \mathfrak{R}$ such that $\mathcal{V}_1 = \mathcal{W}_1$ and \mathcal{V}_2 and \mathcal{W}_2 lie in different M_2 -orbits but in the same H_2 -orbit.

The last step was only necessary in very few of the considered cases, so we mostly could deduce the number of components without calculating a single H -orbit in \mathfrak{X} .

For the results of the calculations we refer to main text, namely Theorem 1.1 and Theorem 5.10.

The reader interested in more technical details concerning the algorithm for the computation of the orbits or in a version of the program should contact us via email, for the script see also [Pe].

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Table 3

$g(F)$	$g(C)$	G	IdSmallGroup	\mathbf{m}	dim	n
3	3	$(\mathbb{Z}_2)^2$ (*)	G(4,2)	$(2^2), (2^6)$	5	1
3	5	$(\mathbb{Z}_2)^3$ (*)	G(8,5)	$(2^2), (2^5)$	4	1
3	5	$\mathbb{Z}_2 \times \mathbb{Z}_4$ (*)	G(8,2)	$(2^2), (2^2, 4^2)$	3	2
3	9	$\mathbb{Z}_2 \times \mathbb{Z}_8$ (*)	G(16,5)	$(2^2), (2, 8^2)$	2	1
3	5	D_4	G(8,3)	$(2^2), (2^2, 4^2)$	3	1
3	7	D_6	G(12,4)	$(2^2), (2^3, 6)$	3	1
3	9	$\mathbb{Z}_2 \times D_4$	G(16,11)	$(2^2), (2^3, 4)$	3	1
3	13	$D_{2,12,5}$	G(24,5)	$(2^2), (2, 4, 12)$	2	1
3	13	$\mathbb{Z}_2 \times A_4$	G(24,13)	$(2^2), (2, 6^2)$	2	1
3	13	S_4	G(24,12)	$(2^2), (3, 4^2)$	2	1
3	17	$\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8)$	G(32,9)	$(2^2), (2, 4, 8)$	2	1
3	25	$\mathbb{Z}_2 \times S_4$	G(48,48)	$(2^2), (2, 4, 6)$	2	1
4	3	S_3	G(6,1)	$(3), (2^6)$	4	1
4	5	D_6	G(12,4)	$(3), (2^5)$	3	1
4	7	$\mathbb{Z}_3 \times S_3$	G(18,3)	$(3), (2^2, 3^2)$	2	2
4	7	$\mathbb{Z}_3 \times S_3$	G(18,3)	$(3), (3, 6^2)$	2	1
4	9	S_4	G(24,12)	$(3), (2^3, 4)$	2	1
4	13	$S_3 \times S_3$	G(36,10)	$(3), (2, 6^2)$	1	1
4	13	$\mathbb{Z}_6 \times S_3$	G(36,12)	$(3), (2, 6^2)$	1	1
4	13	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_3)^2$	G(36,9)	$(3), (2, 4^2)$	1	2
4	21	A_5	G(60,5)	$(3), (2, 5^2)$	1	1

$g(F)$	$g(C)$	G	IdSmallGroup	\mathbf{m}	dim	n
4	25	$\mathbb{Z}_3 \times S_4$	G(72,42)	(3), (2, 3, 12)	1	1
4	41	S_5	G(120,34)	(3), (2, 4, 5)	1	1
5	3	D_4	G(8,3)	(2), (2^6)	4	1
5	4	A_4	G(12,3)	(2), (3^4)	2	2
5	5	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^2$	G(16,3)	(2), ($2^2, 4^2$)	2	3
5	7	$\mathbb{Z}_2 \times A_4$	G(24,13)	(2), ($2^2, 3^2$)	2	2
5	7	$\mathbb{Z}_2 \times A_4$	G(24,13)	(2), ($3, 6^2$)	1	1
5	9	$\mathbb{Z}_8 \ltimes (\mathbb{Z}_2)^2$	G(32,5)	(2), ($2, 8^2$)	1	1
5	9	$\mathbb{Z}_2 \ltimes D_{2,8,5}$	G(32,7)	(2), ($2, 8^2$)	1	1
5	9	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_4 \times \mathbb{Z}_2)$	G(32,2)	(2), (4^3)	1	1
5	9	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^3$	G(32,6)	(2), (4^3)	1	1
5	13	$(\mathbb{Z}_2)^2 \times A_4$	G(48,49)	(2), ($2, 6^2$)	1	1
5	17	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^4$	G(64,32)	(2), ($2, 4, 8$)	1	1
5	21	$\mathbb{Z}_5 \ltimes (\mathbb{Z}_2)^4$	G(80,49)	(2), ($2, 5^2$)	1	2
5	5	$D_{2,8,3}$	G(16,8)	(2^2)	2	1
5	5	$D_{2,8,5}$	G(16,6)	(2^2)	2	2
5	5	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_2)^2$	G(16,3)	(2^2)	2	1

In cases with (*) the dimension of the moduli space and the number of connected components where already given in [P1].

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